THE MANIFESTLY GAUGE INVARIANT MAXWELL-DIRAC EQUATIONS

by

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I contributed 80% of the work in these papers and the other 20% was done by Peter Jarvis.

Lawrence Forbes contributed to 20% and I contributed 80% of the numerical work done in sections (6.1) and (6.2), which is intended to be published at a later date.

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ABSTRACT

We study the Maxwell-Dirac equations, which model the fermionic relativistic electrodynamics in the case where the fermion field is itself the source of the electromagnetic field. This system is formulated by exploiting that fact that the Dirac equation can be algebraically inverted, and the resulting expression for the vector potential in terms of the spinor fields can be directly substituted into Maxwell's equations.

We work in a formalism where the physical states are described by a set of tensor fields, formed from bilinear combinations of (non-Grassmann) spinor fields and Dirac matrices. This results in a set of manifestly gauge invariant equations that lack such unphysical degrees of freedom. Through the use of Fierz expansions on quadratic spinor combinations, and their associated identities, a large set of interrelationships between bilinear fields can be obtained. This permits the description of the Maxwell-Dirac system in terms of tensor current densities, and their quadratic Fierz identities and continuity constraints.

The resulting set of self-coupled Maxwell-Dirac equations is mathematically intractable without further constraint. We show how demanding invariance of the bilinear tensor fields under the action of arbitrary subgroups of the Poincaré group of rotations, translations and boosts reduces the equations to the point where they are more manageable. In this thesis, we demonstrate in detail how the Maxwell-Dirac equations reduce under several example subgroups.

We also develop the gauge invariant bilinear formalism for the stress-energy tensor, which can be used to calculate physical quantities such as the momentum and mass-energy corresponding to a Maxwell-Dirac solution. The calculation is approached from two independent points of view, namely the Belinfante method and the variational method from general relativity, which we find to be in agreement.

Finally, by analogy with the method in electromagnetism, we extend the algebraic inversion of the Dirac equation to the case where the spinors are isospin doublets, and the gauge field corresponds to the non-Abelian group $SU(2)$. Following the definition of non-Abelian bilinears and Fierz identities, the inverted form itself is given formally, with the application of a Neumann series required for an explicit expression.

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- *For Leo, who will always be on top of the world.*

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Chapter 1

Introduction

1.1 The Maxwell-Dirac equations

It is generally understood that the Dirac equation is of central importance in modern physics. The solutions of this equation are four-component objects known as "spinor fields", usually denoted by ψ, and describe the states of relativistic spin-1/2 particles, such as electrons, positrons and quarks. Following the publication of Dirac's seminal paper [13], the close correspondence of the Dirac equation with Nature was confirmed dramatically through its predictions of spectroscopic fine structure, and the experimental discovery of the positron. It was, and remains, a triumph of theoretical physics. The Dirac equation for a free particle is

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0. \tag{1.1}$$

The Dirac equation subsequently went on to play a central role as the exemplar of the matter sector of interacting Abelian and non-Abelian gauge theories in the paradigmatic quantum field theory of particle physics, the Standard Model. The Abelian gauge field here is otherwise known as the electromagnetic vector potential A_μ, and it is introduced by imposing local $U(1)$ gauge covariance on the Dirac equation, whereby the partial derivative term ∂_μ is replaced by the covariant derivative term $D_\mu \equiv \partial_\mu - \mathrm{i}qA_\mu$, where q is the electromagnetic coupling constant. The fact that elements of $U(1)$ commute is what makes it an Abelian group.

The physical behaviour electromagnetic vector potential itself is described by Maxwell's equations, which can be solved in the presence (or absence) of electric charges and currents. That is, a given four-current j^μ distribution will generate an A_μ field, according to the inhomogeneous Maxwell equations. Similarly, in the $U(1)$ gauge covariant Dirac equation, a given externally applied A_μ field will determine the fermion spinor states ψ. The spinor field is related to the four-current by the definition $j^\mu \equiv \overline{\psi}\gamma^\mu\psi$, to be clarified later. A natural self-coupling between these two sets of equations can be made, by considering the case where the spinor field ψ is itself the source of the A_μ field that it interacts with. This is to be contrasted with the "externally applied" A_μ field case, whereby the source of the electromagnetic field lies outside of the distribution in question, and the arbitrarily chosen A_μ configuration determines the form of ψ.

One approach to obtaining such a self-coupled system is to formally "invert" the Dirac equation for the gauge field A_μ, such that it is a function of ψ. From this point of view, it is the form of the spinor field which determines the form of the electromagnetic field, thereby acting as the source. When this form of A_μ as a function of ψ is inserted into the inhomogeneous Maxwell equations, we obtain a set of expressions dependent on ψ only: the self-coupled Maxwell-Dirac equations. With regards to the inversion itself, in an early study [16] pointed out that, the Dirac equation can be written in the form

$$MA_\mu = R_\mu, \tag{1.2}$$

where M is a 4×4 complex matrix, containing only Dirac spinor components. Because $\det M = 0$, the four equations are not linearly independent, and so M is not invertible. However, a more recent study by Booth, Legg and Jarvis [7], which included extensions to higher space-time dimensions, demonstrated that the system of linear equations is indeed invertible if strictly real solutions of the vector potential are required.

The success of the algebraic method for inversion for the Abelian gauge field begs the question as to whether such an algebraic inversion is possible for non-Abelian gauge fields. The Dirac equation itself can be extended to include "internal" degrees of freedom, such as isotopic spin, or "isospin". The simplest example is the Dirac doublet isospinor Ψ, which is composed of two regular four-component Dirac spinors ψ_i, $(i = 1, 2)$. The gauge group $SU(2)$ acts locally on the doublet degree of freedom, producing a unitary transformation on Ψ by rotating in the doublet space by a phase which varies at each point in spacetime. Such transformations do not commute, so $SU(2)$ is a *non-Abelian* gauge group. The Dirac equation for Ψ can be made $SU(2)$ gauge covariant in an analogous way to the $U(1)$ gauge group case. That is, we replace partial derivative terms ∂_μ with the $SU(2)$ covariant derivative $D_\mu \equiv \partial_\mu - (ig/2)\boldsymbol{\tau} \cdot \boldsymbol{W}$, where g is the coupling constant associated with the $SU(2)$ gauge group. The gauge fields $\boldsymbol{W}_{a\mu}$ associated with non-Abelian groups are collectively known as Yang-Mills fields, and are analogous to A_μ. The number of internal degrees of freedom a of a Yang-Mills gauge field is determined by the *adjoint* representation of the gauge group [2].

As we shall demonstrate in chapter 7, the formal inversion can be performed, albeit with some extra non-trivial complications not present in the Abelian case. The interactions in the Standard Model, the electroweak $SU(2)\times U(1)$ and strong $SU(3)$ gauge fields are both non-Abelian, so the extra complications associated with $SU(2)$ should also be present when extending the work done here to other gauge fields.

Before continuing, it is important to note that here we are working with spinor fields which are *non-Grassmann wavefunctions*. That is, ψ is to be considered a semi-classical fermion state analogous to the single component wavefunctions of non-relativistic quantum mechanics, and that the components of these spinor fields *commute*. The necessity for this non-Grassmann treatment of the Maxwell-Dirac equations can be understood by considering the inhomogeneous Maxwell's equations

$$\partial_\nu F^{\nu\mu} = j^\mu, \tag{1.3}$$

where $F^{\nu\mu}$ is the field strength tensor (see section 2.5). In principle, we should be able to take any power of both sides of this equation that we want. However, consider what happens when we take both sides to the power of five. Since ψ_α ($\alpha = 1, 2, 3, 4$) has four components, one for each spinor index α, on the right hand side of the equation each term in a full fifth-order expansion will have at least one repetition of a ψ_α-component. If these components are Grassmann, then by

$$\{\psi_\alpha, \psi_\beta\} = 0, \tag{1.4}$$

everything on the right-hand side vanishes, giving

$$(\partial_\nu F^{\nu\mu})^5 = 0, \tag{1.5}$$

implying that the Maxwell field has no source term, and is free. Given this fundamental contradiction, we must conclude that the correct way to deal with spinor fields in the fully non-linear, non-perturbative Maxwell-Dirac formalism is to treat the spinors as commuting objects.

1.2 The Fierz bilinear formalism

In this thesis, we use a spinor bilinear approach instead of dealing directly with the spinor fields themselves. One example of a spinor bilinear already encountered here is the current density four-vector $j^\mu \equiv \overline{\psi}\gamma^\mu\psi$, which is a tensor field composed from a quadratic, or *bilinear* combination of spinor fields enclosing the 4×4 Dirac matrix γ^μ. Other such bilinears can be produced by replacing the γ^μ by other $4\times$ basis elements of the Dirac-Clifford algebra, or by further altering the spinor fields by charge conjugating, for example see Appendix A.5.

Due to the unphysical and unobservable phase and gauge ambiguities associated with the spinor field, it is often regarded as a secondary entity. Calculations involving measurable quantities remove such ambiguities through various means, such as gauge fixing, or by forming probability density functions through bilinear spinor combinations. There is a substantial literature on the measurability of spinor wavefunctions, with historical origins in the ideas of de Broglie, Bohm, and Pauli. In the case of Dirac theory, we cite Takabayasi [41] as an exponent of the view that indeed the spinor bilinear quantities should be regarded as primary, coordinates of a type of relativistic fluid theory.

The study of the Maxwell-Dirac system in terms of bilinear tensor fields involves the Fierz algebra of quadratic relations between these fields [18]. For example, consider the product of two four-vector bilinears $\overline{\chi}\gamma^\mu\chi \cdot \overline{\psi}\gamma^\nu\psi$. The 4×4 matrix $\chi\overline{\psi}$ can be expanded out in terms of the Dirac-Clifford algebra basis of 4×4 matrices, with bilinears as the basis coefficients (see equation (2.9)). For the $\chi \equiv \psi$ case, there are sixteen real, gauge invariant bilinears, equal to the number of basis elements. Comparing this number with the eight real components of the Dirac spinor, and taking the gauge invariance into account, implies that there must be nine independent algebraic equations [27]. That is, not all of the bilinear fields are independent, but are constrained by these nine equations. The rank-2 tensor fields $s_{\mu\nu}$ and ${}^*s_{\mu\nu}$ can be eliminated from bilinear expressions entirely. Furthermore, the bilinear map is invertible according to the spinor reconstruction theorem by Crawford [10], so given a physical state described by the sixteen bilinear tensor fields, the Dirac spinor field is determined up to a phase.

This bilinear reformulation is generalizable to the non-Abelian gauge field case, which we discuss in chapter 7. For $SU(2)$ doublet isospinors Ψ, matrix combinations such as $\Psi\overline{\Psi}$ can be Fierz expanded in the Pauli matrix basis, including the 2×2 identity τ_0, for the doublet degree of freedom. Due to the tensor product nature of the expansion, for each Pauli matrix, there is a full Dirac-Clifford expansion as in the $\psi\overline{\psi}$ case, such that there are *sixty four* bilinears altogether. Similarly to the Abelian case, there are Fierz identities relating the isodoublet bilinears, and the Lorentz rank-2 fields $S_{i\mu\nu}$ and ${}^{*}S_{i\mu\nu}$ can be eliminated. Incidentally, non-Abelian fluid flow was investigated in [5], where an application to a quark-gluon plasma, involving the $SU(3)$ gauge group, was discussed.

There are many other identities involving the sixteen bilinears beyond the fundamental set [10], which can be derived by using the Fierz expansion along with known identities. The extension of this set of sixteen to other classes of bilinears – such as gauge dependent objects where some of the spinors are charge conjugated (i.e., $\chi \equiv \psi^{\mathrm{c}}$) or contain derivatives so that the spinor field has a Lorentz index – is a necessary part of the mathematical framework for the description of the Maxwell-Dirac equations in bilinear form, as we discuss in chapter 2. A serious attempt at generalizing the fundamental set of nine Fierz identities to include a larger set of bilinear "currents" was presented by Takahashi [42], although bilinears containing spinor derivatives were discussed only briefly.

In chapter 2, we show how given the inverted form of the Dirac equation, one can derive appropriate Fierz identities to remove the explicit appearance of spinor fields entirely, replacing them with bilinear expressions. In this way, the gauge dependent part of the electromagnetic vector potential A_μ can be isolated to a single term containing the *gauge dependent* bilinears m^μ and n^μ, which can then be "picked off", resulting in a manifestly gauge invariant vector potential B_μ. By introducing a tetrad (or vierbein) of bilinear four-vectors, we show that the electromagnetic field strength tensor $F_{\mu\nu}$ can also be written in manifestly gauge invariant bilinear form, clearing the way for the description of the Maxwell-Dirac system in this formalism.

1.3 Symmetry reductions and solutions

The central aim of this thesis is to provide a solid theoretical basis to aid in the search for solutions to the Maxwell-Dirac equations, and to make substantial progress in obtaining such solutions. Although the Fierz bilinear formulation we develop is valid in the general case, experience with other equations in physics suggests that, almost without exception, interesting solutions are those which possess special symmetry properties. Identifying symmetry constraints is therefore likely to be useful in filtering solution types, in addition to aiding in the mathematical tractability of what is an exceptionally difficult problem to solve.

The motivation for obtaining a manifestly gauge invariant formulation is two-fold. As previously discussed, it allows us to write our equations entirely in terms of physical observables. Secondly, it allows us to sidestep questions of gauge potentials which are invariant under spacetime symmetry transformations, thus avoiding extra complication. The constraints arising through the imposition of a certain type of symmetry are simply restrictions on the form that four-vectors can take, as well as what independent variables are allowed. To be more precise, we take our symmetry constraints to be those associated with an *arbitrary* (but fixed) Lie subgroup of the Poincaré group acting on four dimensional Minkowski space-time. All such 158 Poincaré subgroups have been classified up to conjugacy class by Patera, Winternitz and Zassenhaus (hereafter, PWZ) [35]. The tabular presentation of Poincaré subgroups and their associated Lorentz and translation generators can be used as a convenient "shopping list" of symmetry constraints which can be applied to our Maxwell-Dirac system.

In chapters 3 and 4, we demonstrate how the Poincaré subgroups given by PWZ can be applied to yield a reduced set of equations. Our work focuses on the Maxwell-Dirac system, but the technique can in principle be extended to other relativistically covariant theoretical models which are written in terms of scalar and four-vector fields. We choose two standard cases, namely spherical and cylindrical symmetry, as well as two non-standard cases $P_{11,2}$ and $\widetilde{P}_{13,10}$ from [35], which we call the "screw" and "trans-boost" subgroups respectively. The two non-standard cases each have a free continuous parameter, so they in fact represent infinite *families* of symmetries.

The work done in chapter 3 is somewhat technical, and involves the calculation of the forms of scalar and four-vector fields which are invariant under a given symmetry transformation. Deriving these forms requires the use of the Lie derivative, which generally describes the change of a tensor field of given rank along a vector field corresponding to the transformation itself. Setting the Lie derivative to zero is akin to imposing that the tensor field be symmetric under the given transformation. Each Poincaré subgroup we work with has a set of generators associated with it, which themselves have an associated transformation vector field and Lie derivative. Restricting ourselves to the cases where the tensor fields are scalar and four-vector fields, for each given subgroup generator we can use the vanishing Lie derivative to obtain a set of linear partial differential equations, which can be solved via the method of characteristics to obtain the invariant forms corresponding to that generator. This process can be applied cumulatively for successive generators, until we eventually obtain forms for scalar and four-vector fields which are invariant under the action of the entire Poincaré subgroup.

The application of the group invariant forms to the Maxwell-Dirac system is undertaken in chapter 4. Using the restricted forms of σ, ω, j^μ and k^μ, it is shown how the constraining Fierz identities involving these fields, the gauge invariant vector potential B_μ, the field strength tensor $F_{\mu\nu}$, and Maxwell's equations are made relatively simpler. Even with the extra constraints, the resulting systems are still very complex, as can be inferred by observing the forms of the field strength coefficient functions for the cylindrically symmetric case, given explicitly in appendix D.

Despite the ferocious complexity of the Maxwell-Dirac system, there have been many attempts at obtaining solutions in certain highly constrained situations, which we now briefly review. Solitons, or highly localized solutions, have been investigated by Wakano [43] and Lisi [32]. Wakano obtained localized solutions to the Maxwell-

Dirac equations when a dominant electrostatic potential A^0 was assumed, with no solutions existing for the case where A^0 was negligibly small compared with the three-vector potential, A^i. Lisi also numerically obtained an approximate localized solution by neglecting the three-vector potential, arguing that the angular dependence of A^i would break the spherical symmetry. After reintroducing magnetic interaction via a perturbation and finding that it had a small effect on angular dependence, the full Maxwell-Dirac system was considered, and a normalized localized solution was found. Estaban et al. [17] employed a variational approach to finding localized solutions, and proved that stationary solutions do exist without making any approximations to the electromagnetic vector potential.

With regards to Lisi's work, our development of the general spherically symmetric case in chapter 4, in particular the reduction of the field strength tensor in subsection 4.1.3, revealed that in general there is no angular dependence for the magnetic field when the vector potential B_μ is spherically symmetric. What was very surprising, and is a major outcome of our study, was that a *radially* dependent form for the magnetic field is implied by the demand of spherical symmetry itself, in the characteristic form of a magnetic monopole with magnetic charge $q_m = \mp 2\pi/q$. Despite all magnetic terms cancelling from the full self-coupled Maxwell-Dirac system, the presence of the monopole field is a vital component of the spherical symmetry regime, without which it would not exist.

Using the van der Waerden two-spinor formalism, Radford [37] performed a reduction and numerical analysis of the Maxwell-Dirac system under the assumption of a spherically symmetric, static Dirac spinor field, finding that localised compact objects with a shell-like structure exist in this regime. At large distances, a shielding effect from the electrostatic charges dominates, and the field from the central charge distribution approaches a Coulombic form. In a follow-up publication [38], Radford proved a theorem stating that stationary spinors (which translate to static bilinears), subject to weak regularity and decay conditions in the asymptotic region, result in strictly localized Maxwell-Dirac solutions which decay exponentially.

In section 6.1, we undertake our own investigation of the static, spherically symmetric solutions of the Maxwell-Dirac equations, but in the bilinear formalism. We find that the system can be reduced to a single fourth-order, non-linear ODE, for which we show that there is an *exact* (but unphysical) solution, with a large singularity at the origin. After calculating the equilibrium points of the ODE, we find that the only one of physical interest is $\bar{\jmath}_{a,e} = 0$. Linearizing about the system about this point corresponds to a *direct* linearization of the system, yielding the weakly non-linear ODE for the function J exactly the same as for $\bar{\jmath}_a$, but lacking a square root term of ambiguous sign. Using a set of spectral basis functions, we reformulate the problem to the algebraic one of finding a set of Fourier coefficients b_n that minimizes a set of Galerkin residuals R_n. Using Gaussian-form initial guesses, we obtained the two solutions displayed in Figure 6.5. Similarly to the solution obtained by Radford, we find solutions with a static shell-like structure, but for two "orders" with different numbers of shells. In contrast with Radford [37], our solutions lie out from the origin, and are more densely localized.

Intriguingly, Radford [37] also found that imposing static spherical symmetry requires the existence of a magnetic monopole, a finding confirmed by our study, albeit in the more general case where time variation can occur. The requirement of magnetic monopoles is not shared by a subsequent study by Booth and Radford [8] on static cylindrically symmetric solutions. A solution describing a localized Dirac field with a concentric shell structure surrounding a charged axis, as well as finite linear charge density was obtained, and was compared to an equivalent "linearized" system, which lacked the self coupling between the Dirac and Maxwell equations. The relegation of the Maxwell field to an "external" potential resulted in both the localization of the Dirac field and boundedness of the charge density being destroyed.

Das and Kay [11] investigated solutions to the Maxwell-Dirac system where the spinor field was assumed to be the form of a plane wave solution to the free Dirac equation. It was found that non-trivial solutions only exist when $m = 0$, with the additional requirement that associated four vector fields be null. Plane wave solutions were also investigated by Bao and Li [3] as a test case for their broad numerical scheme for the Maxwell-Dirac system, which yielded exact results. On the technical matter of the existence of solutions in general, a very formidable global existence proof to the Cauchy problem for the Maxwell-Dirac equations has been provided by Flato et al. [19], extending upon previous work on the matter by Gross [23].

We should also give an honourable mention to the unpublished work of Legg [31], upon which our current work is inspired, where Poincaré subgroup invariant solutions of the manifestly gauge invariant Maxwell-Dirac equations were investigated. Focusing on transitive Poincaré subgroups that have four-dimensional orbits, Legg found that the only one that resulted in a physically interesting solution was $\widetilde{P}_{13,10}$, what we call the "trans-boost" subgroup, and a reduction of the Maxwell-Dirac system and closed form solution implying hyperbolic distributions as in Figures 6.10 and 6.11 was subsequently presented. Our work in section 6.3 constitutes an independent, alternative confirmation of Legg's result, with the insight that it corresponds to a *special case* in a class of more general solutions, the parametrization of which we discuss in some detail.

1.4 The bilinear stress-energy tensor

When choosing a mathematical construct to model the stress-energy of a given physical system, it appears after investigation that the situation is not entirely straightforward. Likely, the first model found during any search of the literature is the so-called "canonical form" [22], which is defined as the Noether symmetry current associated with invariance of the Lagrangian density under space-time translations. However, the canonical form has the unfortunate drawback of not being either symmetric in its two indices, or gauge invariant.

Many attempts have been made to rectify these problems, two of the most prominent being the Belinfante form [4], and the variational form from general relativity [44]. The basis of the Belinfante approach is to extend the invariance of the Lagrangian to include contributions from the Lorentz transformations, so that the Noether symmetry current becomes that associated with the full Poincaré group. In this way, the canonical term is symmetrized [45], and extra "correction" terms are present, which are attributed to the spin contribution to the stress-energy.

The variational approach uses the action principle to relate the variation of the Hilbert action of space-time to that of matter, then invoking Einstein's equations to identify the matter part to the stress-energy tensor. The result is an expression for the stress-energy which is proportional to the functional derivative of the action of matter, with respect to the inverse metric. The presence of the metric ensures that the stress-energy tensor is manifestly symmetric.

Regardless of the independent nature of their derivation, Goedecke pointed out [20] that in the limit of flat space-time, the Belinfante and variational forms of the stress-energy tensor must agree. The equivalence in the integral spin field case was proven by Rosenfeld [40], and Goedecke provided evidence for equivalence in the half-integral spin field case via a series of examples, but was not able to provide a general proof. Such a proof for the half-integral case was published shortly afterwards by Lord [33], using the vierbein formalism.

Much effort has been made to "improve" the stress-energy tensor, either by generalizing it beyond the Belinfante/variational forms, or by altering it so that it is compatible with a given theory. An example of the generalization aspect is the work done by Gotay and Marsden, who model the stress-energy in terms of fluxes of the multimomentum map across space-time hypersurfaces [21]. The Gotay-Marsden stress-energy tensor naturally includes the spin "correction terms" present in the Belinfante formula, but in a more generalized fashion, as well as coinciding with the variational form in the presence of a space-time metric. Work has also been done by Callan, Coleman and Jackiw on making the cut-off dependent symmetric stress-energy tensor compatible with renormalized perturbation theory, by constructing appropriate counter terms in order to make it finite at arbitrarily large cut-off values [9]. The renormalization compatible stress-energy tensor is also compatible with an altered, but phenomenologically consistent, version of general relativity, and simplifies the currents associated with scale and conformal transformations in which the stress-energy appears.

The work we present in chapter 5 can be viewed as analogous to one of Goedecke's examples, namely the coupled Maxwell-Dirac fields, but in an alternate formalism where the spinor fields are mapped bilinearly to a set of tensor fields.

The derivation of the bilinear form of the Maxwell-Dirac-Belinfante tensor is undertaken in section 5.1. Following a brief derivation of the Belinfante tensor for a free Dirac particle in the spinor representation, we derive the Fierz identity which allows us to express the spinor-dependent Belinfante tensor exclusively in terms of bilinears. A more detailed version of this derivation is relegated to appendix E. The known tensorial forms of the electromagnetic interaction and Maxwell field stress-energies are then added to the free Dirac contribution, resulting in a manifestly symmetric and gauge independent bilinear form of the Maxwell-Dirac-Belinfante tensor.

Section 5.2 presents an independent derivation of the Maxwell-Dirac stress-energy tensor, which in this case uses the variational form known from general relativity. Beginning with the Lagrangian density for and electromagnetically interacting Dirac particle, and initially ignoring the Maxwell field contribution since we are mainly interested in the behaviour of the bilinear Dirac contribution, we convert it to its analogous bilinear form, using a contracted form of the Fierz identity obtained in section 5.1.2. A brief review of how the variational stress-energy is obtained is then given. Then, using the general relativistically covariant form of the bilinear Dirac Lagrangian, the variational stress-energy is obtained, and is found to be in agreement with the Maxwell-Dirac-Belinfante tensor.

In section 5.3, the restrictions imposed by one of the example Poincaré symmetry subgroups, namely the spherical symmetry group $SO(3)$, is applied to the bilinear form of the Belinfante tensor. Following the discussion of Maxwell-Dirac solutions under static spherical symmetry in section 6.1, the spherical stress-energy reduction is extended to the static case in section 6.2, so that it can be applied to the obtained solution forms. The remainder of this section discusses how total mass and charge is calculated from T^{00} and $\bar{j}_a$ respectively, as well as the technical complications arising from the actual calculations using the numerical data corresponding to the single-hump $\bar{j}_a$ solution.

Chapter 2

Maxwell-Dirac Theory and Gauge Invariant Formulation

We now introduce the theoretical content of this thesis, by showing explicitly how the Maxwell-Dirac equations are formulated in terms of bilinear tensor fields. We begin by demonstrating the inversion of the $U(1)$ gauge covariant Dirac equation for the electromagnetic vector potential. It should be understood that the concerns raised by Eliezer regarding the invertibility of the Dirac equation [16] have been addressed by Booth, Legg and Jarvis [7] by regarding the gauge field as a real object. There is an alternative inverted form of the gauge field, which follows by an analogous calculation involving dual objects, but we do not present this calculation explicitly here. It is important to note however, that we have found from experience that both dual (containing γ_5 terms) and their non-dual counterparts have to be considered during "bilinearization", whereby spinor-dependent expressions are bilinearly mapped to the analogous representative set of tensor fields.

The Fierz expansion, a concept central to the obtaining of bilinear expressions from quadratic 4×4 spinor objects, such as $\psi_\alpha \overline{\chi}_\beta$, is presented. The Dirac spinor indices α and β both run from 1 to 4, and denote a degree of freedom distinct from Minkowski spacetime. The process for obtaining the bilinearized form of the Dirac equation is then outlined, which primarily involves the derivation of appropriate Fierz identities to replace the explicitly spinor-dependent objects appearing in A^μ. A major advantage of the bilinear approach then becomes apparent, whereby the gauge-dependence of the gauge field A^μ is distilled into a single term, which can be removed by defining a new gauge-independent field B^μ.

What comes next is the essence of the self-coupled nature of the Maxwell-Dirac system, in that we insert the bilinear expression for A^μ, the inverted Dirac equation, into the electromagnetic field strength tensor $F_{\mu\nu}$, which itself represents the electromagnetic field in the relativistically covariant form of Maxwell's equations. It is well known that $F_{\mu\nu}$, the "four-curl" of A^μ is a gauge invariant object, but we can take a step further, and rewrite it as a *manifestly* gauge invariant object, in which only terms understood to be gauge independent explicitly appear. This can be done by defining the tetrad, or "*vierbein*", of four four-vector bilinears, and using a known Fierz identity to eliminate the gauge-dependent terms. We conclude the chapter with a summary of the equations constituting the Maxwell-Dirac system in manifestly gauge invariant bilinear form.

2.1 Dirac equation inversion

The Dirac equation is the relativistic wave equation for spin-1/2 particles, such as electrons. For fermionic particles of charge q interacting with an electromagnetic field, we require solutions to the Dirac equation form-invariant under a $U(1)$ Abelian gauge transformation, given by

$$(\mathrm{i}\gamma^\nu\partial_\nu - q\gamma^\nu A_\nu - m)\psi = 0. \tag{2.1}$$

Conventions for Dirac-Clifford algebra and spinor manipulations are given in appendix A. Our goal is to isolate the vector potential A^μ. Rearranging gives us

$$\gamma^\nu\psi A_\nu = q^{-1}(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi. \tag{2.2}$$

We can form a bilinear spinor expression by multiplying by $\overline{\psi}\gamma^\mu$ from the left. Using the Dirac identity (A.10), our expression becomes

$$\overline{\psi}\psi A^\mu - \mathrm{i}\overline{\psi}\sigma^{\mu\nu}\psi A_\nu = q^{-1}[\mathrm{i}\overline{\psi}\gamma^\mu\gamma^\nu(\partial_\nu\psi) - m\overline{\psi}\gamma^\mu\psi]. \tag{2.3}$$

In order to eliminate the second term on the left-hand side, turn to the charge conjugate Dirac equation, which is similar in form to (2.1), but with the sign of the charge reversed:

$$(\mathrm{i}\gamma^\nu\partial_\nu + q\gamma^\nu A_\nu - m)\psi^{\mathrm{c}} = 0. \tag{2.4}$$

The charge conjugate spinor is defined in terms of the regular spinor as [26]

$$\psi^{\mathrm{c}} = C\overline{\psi}^{\mathrm{T}} = \mathrm{i}\gamma^2\gamma^0\overline{\psi}^{\mathrm{T}}. \tag{2.5}$$

Similarly rearranging and left-multiplying by $\overline{\psi}^{\mathrm{c}}\gamma^\mu$, then applying the appropriate charge conjugation identities in appendix A.4, gives

$$-\overline{\psi}\psi A^\mu - \mathrm{i}\overline{\psi}\sigma^{\mu\nu}\psi A_\nu = q^{-1}[\mathrm{i}(\partial_\nu\overline{\psi})\gamma^\nu\gamma^\mu\psi + m\overline{\psi}\gamma^\mu\psi]. \tag{2.6}$$

Subtracting (2.6) from (2.3) and again using (A.10), gives us the inverted form of the Dirac equation

$$A^\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}(\partial^\mu\psi) - (\partial^\mu\overline{\psi})\psi] + \partial_\nu s^{\mu\nu} - 2mj^\mu}{\sigma}, \tag{2.7}$$

where we have used the shorthand notation for Dirac bilinear tensors, listed in appendix A.5. There is an alternative inverted form for the Dirac equation, which involves left-multiplication of (2.1) by $\overline{\psi}\gamma_5\gamma^\mu$ to form bilinears. Following the same steps as above yields the expression

$$A^\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}\gamma_5(\partial^\mu\psi) - (\partial^\mu\overline{\psi})\gamma_5\psi] + \partial_\nu {}^*s^{\mu\nu}}{\omega}, \tag{2.8}$$

which lacks a mass-dependent term. In addition to these inversions, we can derive other expressions by left-multiplying (2.2) and its charge conjugate analogue by $\overline{\psi}\Gamma$ and $\overline{\psi}^c\Gamma$ respectively, for general elements Γ of the Dirac-Clifford algebra, then adding or subtracting the two equations. Among the resulting expressions are the continuity equation $\partial_\mu j^\mu = 0$ and the current-field coupling $j^\nu A_\nu$. The full list of expressions obtained from "bilinearizing" the Dirac equation is given in appendix B.

2.2 Fierz identities

It is well known that quadratic relationships between Dirac bilinears of the form $\overline{\chi}\Gamma_R\psi$, where Γ_R represents the sixteen basis elements of the Dirac-Clifford algebra $\Gamma_R = \{I, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\gamma^\mu, \gamma_5\}$, can be derived via a Fierz expansion of the product of two Dirac spinors in this basis

$$\begin{aligned}\psi\overline{\chi} = \sum_{R=1}^{16} a_R\Gamma_R = (1/4)(\overline{\chi}\psi)I + (1/4)(\overline{\chi}\gamma_\mu\psi)\gamma^\mu + (1/8)(\overline{\chi}\sigma_{\mu\nu}\psi)\sigma^{\mu\nu} \\ - (1/4)(\overline{\chi}\gamma_5\gamma_\mu\psi)\gamma_5\gamma^\mu + (1/4)(\overline{\chi}\gamma_5\psi)\gamma_5. \end{aligned} \tag{2.9}$$

Here, a_R are the numerical coefficients multiplied by the Dirac bilinears. This expansion is inserted into products such as $j^\mu k^\nu \equiv \overline{\psi}\gamma^\mu(\psi\overline{\psi})\gamma_5\gamma^\nu\psi$, for example. Experimenting with different combinations of bilinears, and combining the resulting equations yields many different interrelationships. Many Fierz identities are summarized in [42] and [10], but the most important for our purposes are

$$j_\nu j^\nu = -k_\nu k^\nu = -m^\nu m_\nu = -n^\nu n_\nu = \sigma^2 - \omega^2, \tag{2.10}$$

$$j_\nu k^\nu = j_\nu m^\nu = j_\nu n^\nu = k_\nu m^\nu = k_\nu n^\nu = m_\nu n^\nu = 0, \tag{2.11}$$

$$\epsilon_{\mu\nu\rho\sigma} j^\rho k^\sigma = m_\mu n_\nu - m_\nu n_\mu, \tag{2.12}$$

$$s_{\mu\nu} = \frac{(\sigma\epsilon_{\mu\nu}{}^{\rho\sigma} - \omega\delta_{\mu\nu}{}^{\rho\sigma})j_\rho k_\sigma}{\sigma^2 - \omega^2}, \tag{2.13}$$

$$^*s_{\mu\nu} = \frac{(\omega\epsilon_{\mu\nu}{}^{\rho\sigma} - \sigma\delta_{\mu\nu}{}^{\rho\sigma})j_\rho k_\sigma}{\sigma^2 - \omega^2}. \tag{2.14}$$

This method can be extended to the $SU(2)$ spinor doublet case by building into (2.9) an expansion over the Pauli matrices, including the 2×2 identity matrix. Analogous non-Abelian expressions to (2.13) and (2.14) are derived in section 7.2 via the use of such expansions.

2.3 Vector potential

In order to avoid arbitrarily fixing the gauge, here we eliminate gauge dependent terms from our equations entirely, so that our Maxwell-Dirac system is *manifestly* gauge invariant. We do this by reformulating (2.7) and (2.8) to be entirely in terms of the bilinear tensors listed in appendix A.5, getting rid of the incongruous

$[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]$ terms. We can then easily pick off the gauge dependent parts (m^μ and n^μ), to define a gauge-invariant vector potential, which we denote B^μ. A Maxwell-Dirac formalism, completely in terms of manifestly gauge invariant tensors is then derived.

This approach is in the spirit of Takabayasi [41], whose philosophy regarded a relativistic quantum mechanical formalism strictly involving only "observables", such as tensors, as being preferable to one where somewhat "unphysical" objects such as spinors are explicitly included. A detailed derivation of B^μ is given in appendix C, but we give a brief overview here.

First, take the sum of the two versions of the inverted Dirac equation (2.7), (2.8), and divide by 2

$$A_\mu = \frac{1}{4q}\left\{\frac{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega + \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma}{\sigma\omega} + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega} \right.$$
$$\left. - \frac{2mj_\mu}{\sigma}\right\}. \tag{2.15}$$

The appropriate tensor forms needed to replace the spinor terms are $j^\nu(\partial_\mu k_\nu)$ and $m^\nu(\partial_\mu n_\nu)$. Consider the first tensor:

$$j^\nu(\partial_\mu k_\nu) = \overline{\psi}\gamma^\nu\psi\cdot(\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi + \overline{\psi}\gamma^\nu\psi\cdot\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi). \tag{2.16}$$

Fierz expanding both terms and rearranging gives

$$j^\nu(\partial_\mu k_\nu) = (2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma - (1/3)k^\nu(\partial_\mu j_\nu). \tag{2.17}$$

We must also consider the Fierz expansion of $k^\nu(\partial_\mu j_\nu)$ in order to eliminate it from the expression, which after rearrangement is given by

$$k^\nu(\partial_\mu j_\nu) = (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma - (2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - (1/3)j^\nu(\partial_\mu k_\nu). \tag{2.18}$$

Using these two equations yields the new Fierz identity

$$j^\nu(\partial_\mu k_\nu) = -k^\nu(\partial_\mu j_\nu) = [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma. \tag{2.19}$$

In order to eliminate both of the bracketed spinor terms, we require another independent expression involving them. Such an expression is provided by $m^\nu(\partial_\mu n_\nu)$, which in spinor form is

$$\begin{aligned} m^\nu(\partial_\mu n_\nu) = (\mathrm{i}/4)[&\overline{\psi}^{\mathrm{c}}\gamma^\nu\psi\cdot(\partial_\mu\overline{\psi})\gamma_\nu\psi^{\mathrm{c}} + \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi\cdot\overline{\psi}\gamma_\nu(\partial_\mu\psi^{\mathrm{c}}) - \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi\cdot(\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\nu\psi \\ &- \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi\cdot\overline{\psi}^{\mathrm{c}}\gamma_\nu(\partial_\mu\psi) + \overline{\psi}\gamma^\nu\psi^{\mathrm{c}}\cdot(\partial_\mu\overline{\psi})\gamma_\nu\psi^{\mathrm{c}} + \overline{\psi}\gamma^\nu\psi^{\mathrm{c}}\cdot\overline{\psi}\gamma_\nu(\partial_\mu\psi^{\mathrm{c}}) \\ &- \overline{\psi}\gamma^\nu\psi^{\mathrm{c}}\cdot(\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\nu\psi - \overline{\psi}\gamma^\nu\psi^{\mathrm{c}}\cdot\overline{\psi}^{\mathrm{c}}\gamma_\nu(\partial_\mu\psi)]. \end{aligned} \tag{2.20}$$

Fierz expanding the individual terms, and applying the appropriate charge conjugate and complex conjugate bilinear identities from appendix A, we obtain

$$\begin{aligned} m^\nu(\partial_\mu n_\nu) = (\mathrm{i}/4)\{&2[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - 2[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega \\ &+ [\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu - [\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu\}. \end{aligned} \tag{2.21}$$

Performing another round of Fierz expansions on the last two terms eventually provides us with another Fierz identity

$$
\begin{aligned}
[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu &= -[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu \\
&= [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega, \qquad (2.22)
\end{aligned}
$$

which gives us our desired identity

$$
m^\nu(\partial_\mu n_\nu) = \mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega. \qquad (2.23)
$$

It can be shown via a similar process that

$$
m^\nu(\partial_\mu n_\nu) = -n^\nu(\partial_\mu m_\nu). \qquad (2.24)
$$

From substitution and rearrangement of (2.19) and (2.23), we get the expressions needed to eliminate spinors from the inverted Dirac equation entirely

$$
\begin{aligned}
[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] &= -(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\omega + \mathrm{i}m^\nu(\partial_\mu n_\nu)\sigma], \qquad (2.25)\\
[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] &= -(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\sigma + \mathrm{i}m^\nu(\partial_\mu n_\nu)\omega]. \qquad (2.26)
\end{aligned}
$$

Combining these two identities in the form they appear in (2.15), we get

$$
\begin{aligned}
&\{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega + \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma\}(\sigma\omega)^{-1} \\
&\quad = \frac{2m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \frac{\mathrm{i}j^\nu(\partial_\mu k_\nu)}{\sigma^2-\omega^2}\left[\frac{\sigma^2+\omega^2}{\sigma\omega}\right]. \qquad (2.27)
\end{aligned}
$$

Substituting into (2.15), we obtain an expression for A_μ exclusively in tensor form

$$
A_\mu = \frac{1}{4q}\left\{\frac{2m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\sigma^2+\omega^2}{\sigma\omega(\sigma^2-\omega^2)}\right] + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega} - \frac{2mj_\mu}{\sigma}\right\}. \qquad (2.28)
$$

We can improve on this by substituting (2.25) and (2.26) into (2.7) and (2.8) respectively, then subtracting and rearranging, obtaining a consistency condition for the Dirac equation in tensor form

$$
\mathrm{i}j^\nu(\partial_\mu k_\nu) = 2m\omega j_\mu + \sigma\partial_\nu {}^*s_\mu{}^\nu - \omega\partial_\nu s_\mu{}^\nu. \qquad (2.29)
$$

Substituting this into (2.28), we obtain after some algebraic manipulation, the final form of the inverted Dirac equation in tensor form

$$
A_\mu = \frac{1}{2q}\frac{m^\nu(\partial_\mu n_\nu) + \sigma\partial_\nu s_\mu{}^\nu - \omega\partial_\nu {}^*s_\mu{}^\nu - 2m\sigma j_\mu}{\sigma^2-\omega^2}. \qquad (2.30)
$$

We define the gauge invariant vector potential simply by subtracting the only gauge dependent part from A_μ

$$
B_\mu = A_\mu - \frac{1}{2q}\frac{m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} = \frac{1}{2q}\frac{\sigma\partial_\nu s_\mu{}^\nu - \omega\partial_\nu {}^*s_\mu{}^\nu - 2m\sigma j_\mu}{\sigma^2-\omega^2}. \qquad (2.31)
$$

The Fierz identities (2.13) and (2.14) can be used to eliminate the rank-2 tensors from the B_μ expression entirely. With a small amount of work, we find that

$$
B_\mu = (1/2q)\{\epsilon_\mu{}^{\nu\rho\sigma}[(\sigma^2-\omega^2)\partial_\nu(j_\rho k_\sigma) - (1/2)j_\rho k_\sigma\partial_\nu(\sigma^2-\omega^2)]
$$

$$+\,\delta_\mu{}^{\nu\rho\sigma}[(\partial_\nu\sigma)\omega-\sigma(\partial_\nu\omega)]j_\rho k_\sigma\}(\sigma^2-\omega^2)^{-2}-(1/q)m\sigma j_\mu(\sigma^2-\omega^2)^{-1}. \quad (2.32)$$

It is apparent that B_μ is only finite when $\sigma^2-\omega^2\neq 0$. It is perhaps appropriate to mention here that a common alternative definition of the pseudoscalar bilinear [10], [41] is $\varpi=\overline{\psi}\mathrm{i}\gamma_5\psi$, so performing a change of variables, we would have in the denominator $\sigma^2+\varpi^2$. Since ϖ is real [10], implying that ω is purely imaginary, the denominator only vanishes for σ and ω vanishing independently. Additionally, we have the condition that $\sigma^2-\omega^2\geq 0$.

2.4 The tetrad of bilinears

Here we make the claim based on (2.10) and (2.11) that the four mutually orthogonal four vector fields j^μ, m^μ, n^μ and k^μ constitute the columns of a tetrad [31]

$$\mathrm{t}^\mu{}_\alpha=(\sigma^2-\omega^2)^{-1/2}[j^\mu,m^\mu,n^\mu,k^\mu]. \quad (2.33)$$

$\mu=0,1,2,3$ is the spacetime index as usual, and $\alpha=0,1,2,3$ labels the columns, with $\alpha=0$ denoting the timelike field j^μ and $\alpha=1,2,3$ denoting the spacelike fields, m^μ, n^μ and k^μ respectively. Gauge transformations can be thought of as rotations in the $m^\mu-n^\mu$ plane. The coefficient $(\sigma^2-\omega^2)^{-1/2}$ behaves as a normalizing factor. Now consider the contraction of two tetrads via the μ index

$$\mathrm{t}^\alpha{}_\mu\mathrm{t}^\mu{}_\beta=(\sigma^2-\omega^2)^{-1}\begin{pmatrix} j_\mu j^\mu & j_\mu m^\mu & j_\mu n^\mu & j_\mu k^\mu\\ -m_\mu j^\mu & -m_\mu m^\mu & -m_\mu n^\mu & -m_\mu k^\mu\\ -n_\mu j^\mu & -n_\mu m^\mu & -n_\mu n^\mu & -n_\mu k^\mu\\ -k_\mu j^\mu & -k_\mu m^\mu & -k_\mu n^\mu & -k_\mu k^\mu\end{pmatrix}=\delta^\alpha{}_\beta, \quad (2.34)$$

which in matrix notation is simply $(\eta\mathrm{t}^{\mathrm{T}}\eta)\mathrm{t}=I$, implying that $(\eta\mathrm{t}^{\mathrm{T}}\eta)=\mathrm{t}^{-1}$. Putting the inverse tetrad on the right, and labeling the indices appropriately gives us

$$\mathrm{t}^\mu{}_\alpha\mathrm{t}^\alpha{}_\nu=(\sigma^2-\omega^2)^{-1}(j^\mu j_\nu-m^\mu m_\nu-n^\mu n_\nu-k^\mu k_\nu)=\delta^\mu{}_\nu, \quad (2.35)$$

an identity which we will find useful in the next section. Taking the derivative of (2.34) and rearranging gives

$$\mathrm{t}^\nu{}_\alpha(\partial_\mu\mathrm{t}_{\nu\beta})=-\mathrm{t}^\nu{}_\beta(\partial_\mu\mathrm{t}_{\nu\alpha}), \quad (2.36)$$

which is antisymmetric in α and β. In fact, this is a generalization of (2.19) and (2.24), which were originally derived via the Fierz expansion method. A result of the antisymmetry is that if $\alpha=\beta$, the term vanishes, which tells us that the four vector fields multiplied by the normalizing factor $(\sigma^2-\omega^2)^{-1}$ are orthogonal to their own partial four derivatives. Substituting the components of the tetrad for $\alpha=\beta$ into (2.36) gives an identity in terms of the unnormalized vectors

$$j^\nu(\partial_\mu j_\nu)=-m^\nu(\partial_\mu m_\nu)=-n^\nu(\partial_\mu n_\nu)=-k^\nu(\partial_\mu k_\nu)=\sigma(\partial_\mu\sigma)-\omega(\partial_\mu\omega), \quad (2.37)$$

which is just the derivative of (2.10).

2.5 Field strength tensor

The electromagnetic field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.38}$$

This is a gauge invariant tensor, but in this form it is not *manifestly* gauge invariant because it explicitly contains the gauge dependent term A_μ. The manifestly gauge invariant $F_{\mu\nu}$ was originally obtained by Takabayasi [41], then again by Legg [31], this derivation mirroring that of the latter. Replace A_μ using (2.31)

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{2q}\left\{\partial_\mu\left[\frac{m^\rho(\partial_\nu n_\rho)}{\sigma^2-\omega^2}\right] - \partial_\nu\left[\frac{m^\rho(\partial_\mu n_\rho)}{\sigma^2-\omega^2}\right]\right\}. \tag{2.39}$$

Our goal is to eliminate the gauge dependent terms m^μ and n^μ from the expression entirely. Expanding the derivatives in the bracketed term gives us

$$\begin{aligned}\partial_\mu\left[\frac{m^\rho(\partial_\nu n_\rho)}{\sigma^2-\omega^2}\right] - \partial_\nu\left[\frac{m^\rho(\partial_\mu n_\rho)}{\sigma^2-\omega^2}\right] &= \frac{(\partial_\mu m^\rho)(\partial_\nu n_\rho) - (\partial_\nu m^\rho)(\partial_\mu n_\rho)}{\sigma^2-\omega^2} \\ &+ \frac{m^\rho(\partial_\mu n_\rho)\partial_\nu(\sigma^2-\omega^2) - m^\rho(\partial_\nu n_\rho)\partial_\mu(\sigma^2-\omega^2)}{(\sigma^2-\omega^2)^2}.\end{aligned} \tag{2.40}$$

Focusing on the left-hand term, if we insert the identity $\delta_\sigma{}^\rho$ between each four vector derivative in the numerator, then expand using (2.35), we get

$$\begin{aligned}&[(\partial_\mu m^\sigma)\delta_\sigma{}^\rho(\partial_\nu n_\rho) - (\partial_\nu m^\sigma)\delta_\sigma{}^\rho(\partial_\mu n_\rho)](\sigma^2-\omega^2)^{-1} \\ &= [(\partial_\mu m^\sigma)j_\sigma j^\rho(\partial_\nu n_\rho) - (\partial_\mu m^\sigma)k_\sigma k^\rho(\partial_\nu n_\rho) - (\partial_\mu m^\sigma)m_\sigma m^\rho(\partial_\nu n_\rho) \\ &\quad - (\partial_\mu m^\sigma)n_\sigma n^\rho(\partial_\nu n_\rho) - (\partial_\nu m^\sigma)j_\sigma j^\rho(\partial_\mu n_\rho) + (\partial_\nu m^\sigma)k_\sigma k^\rho(\partial_\mu n_\rho) \\ &\quad + (\partial_\nu m^\sigma)m_\sigma m^\rho(\partial_\mu n_\rho) + (\partial_\nu m^\sigma)n_\sigma n^\rho(\partial_\mu n_\rho)](\sigma^2-\omega^2)^{-2} \\ &= (m^\sigma n^\rho - m^\rho n^\sigma)[(\partial_\mu j_\sigma)(\partial_\nu j_\rho) - (\partial_\mu k_\sigma)(\partial_\nu k_\rho)](\sigma^2-\omega^2)^{-2} \\ &\quad - [m^\sigma(\partial_\mu n_\sigma)\partial_\nu(\sigma^2-\omega^2) - m^\sigma(\partial_\nu n_\sigma)\partial_\mu(\sigma^2-\omega^2)](\sigma^2-\omega^2)^{-2}.\end{aligned} \tag{2.41}$$

To get to the first term in the last step, we used the tetrad identity (2.36) to switch the partial derivatives onto the gauge independent tensors, then factorized. The second term in the last step follows from using (2.37) to replace terms like $m^\rho(\partial_\mu m_\rho)$ with $-(1/2)\partial_\mu(\sigma^2-\omega^2)$, and using and (2.36) to place all the derivatives onto the n_σ vectors. Substituting (2.41) into (2.40), the right-hand term in (2.40) cancels out, leaving us with

$$\partial_\mu\left[\frac{m^\rho(\partial_\nu n_\rho)}{\sigma^2-\omega^2}\right] - \partial_\nu\left[\frac{m^\rho(\partial_\mu n_\rho)}{\sigma^2-\omega^2}\right] = \frac{(m^\sigma n^\rho - m^\rho n^\sigma)[(\partial_\mu j_\sigma)(\partial_\nu j_\rho) - (\partial_\mu k_\sigma)(\partial_\nu k_\rho)]}{(\sigma^2-\omega^2)^2}. \tag{2.42}$$

Applying the Fierz identity (2.12), we can eliminate the gauge dependent tensors entirely, giving us the desired expression

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{2q}\frac{\epsilon^{\sigma\rho\kappa\tau}j_\kappa k_\tau[(\partial_\mu j_\sigma)(\partial_\nu j_\rho) - (\partial_\mu k_\sigma)(\partial_\nu k_\rho)]}{(\sigma^2-\omega^2)^2}, \tag{2.43}$$

the manifestly gauge invariant electromagnetic field strength tensor.

2.6 Maxwell-Dirac equations

In summary, our Maxwell-Dirac system consists of the Fierz identities

$$j_\mu j^\mu = -k_\mu k^\mu = \sigma^2 - \omega^2, \tag{2.44}$$

$$j_\mu k^\mu = 0, \tag{2.45}$$

the gauge invariant form of the inverted Dirac equation

$$\begin{aligned} B_\mu = (1/2q)\{&\epsilon_\mu{}^{\nu\rho\sigma}[(\sigma^2-\omega^2)\partial_\nu(j_\rho k_\sigma) - (1/2)j_\rho k_\sigma \partial_\nu(\sigma^2-\omega^2)] \\ &+ \delta_\mu{}^{\nu\rho\sigma}[(\partial_\nu\sigma)\omega - \sigma(\partial_\nu\omega)]j_\rho k_\sigma\}(\sigma^2-\omega^2)^{-2} - (1/q)m\sigma j_\mu(\sigma^2-\omega^2)^{-1}, \end{aligned} \tag{2.46}$$

the manifestly gauge invariant field strength tensor

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{2q}\frac{\epsilon^{\sigma\rho\kappa\tau} j_\kappa k_\tau[(\partial_\mu j_\sigma)(\partial_\nu j_\rho) - (\partial_\mu k_\sigma)(\partial_\nu k_\rho)]}{(\sigma^2-\omega^2)^2}, \tag{2.47}$$

and the inhomogeneous Maxwell equations

$$\partial_\nu F^{\nu\mu} = q j^\mu. \tag{2.48}$$

We can also use two of the physical constraints obtained from manipulating the Dirac equation listed in appendix B, the continuity equation and its pseudovector analogue

$$\partial_\mu j^\mu = 0, \tag{2.49}$$

$$\partial_\mu k^\mu = -2\mathrm{i}m\omega. \tag{2.50}$$

Because we are using the inverted Dirac equation in the field strength tensor instead of an "external" electromagnetic field, we are demanding that the charged fermionic field itself be the source of the field. This does not preclude the addition of electromagnetic fields from external sources, but our intention is to study the physical behaviour of Dirac matter fields under the influence of their internal Maxwell fields. However, as we shall see in section 4.1.3, imposing some symmetries requires the presence of an external Maxwell field, despite their implicit exclusion. In the spherical symmetry case, the external field is the magnetic monopole, a solution of the vacuum Maxwell equations in the case where the origin is an excluded point ($r \neq 0$).

Substituting the vector potential into the Maxwell equations yields a self-consistent set of PDEs describing the behaviour of the fermion field under its own electromagnetic field. In the next chapters, we will consider how the imposition of symmetry under select subgroups of the Poincaré group affects this system, and as we shall see, depending on the subgroup we choose, the complexity of the Maxwell-Dirac system varies dramatically.

Chapter 3

Poincaré Subgroups and Invariant Tensor Forms

Now that we have obtained our Maxwell-Dirac system, we must consider how we are to go about finding symmetry reductions. Observing equations (2.46)-(2.48), we can see that we have a third-order non-linear coupled set of PDEs; a very formidable system indeed. It is fortunate that most physically interesting situations have symmetry under a certain subgroup of the Poincaré group, two prime examples being the spherical and cylindrical symmetries. These particular cases were studied by Radford and Booth [37], [8], with the additional constraint of having a static Dirac field, which assumes that there must exist a Lorentz frame in which there is no current flow, $j^\mu = \delta_0{}^\mu j^0$. Since their work was done in the gauge dependent two-spinor formalism, a specific gauge was chosen to remove gauge ambiguity. In this study, we will apply these same symmetries to the Maxwell-Dirac system, but since we are working with inherently gauge invariant tensor fields only, we have the advantage of not having to choose any specific gauge arbitrarily, which could result in a loss of generality.

The situations of spherical and cylindrical symmetry are but two of many possibilities for analyzing the structure of the reduced Maxwell-Dirac system in the presence of symmetries. Given that we are dealing with relativistic wave equations compatible with the underlying action of the Poincaré group of transformations on Minkowski space, appropriate symmetries are therefore subgroups of the Poincaré group. The comprehensive classification by Patera, Winternitz and Zassenhaus [35] (hereafter PWZ), identifies all 158 continuous subgroups of the Poincaré group up to conjugacy, and the methods we develop are in principle able to give Maxwell-Dirac symmetry reductions for *any* of these subgroups. At the Lie algebra level, the PWZ scheme uses the known list F_i, $i = 1, 2, ..., 15$ of distinct subalgebras of the Lorentz group Lie algebra, to establish a corresponding classification $P_{i,j}$ of Poincaré subalgebras, where the Lorentz part F_i is extended by an ideal $N_{i,j}$ containing translation generators for some $j = 1, 2, ..., n_i$. In addition, there exists a further exceptional set denoted $\widetilde{P}_{i,j}$, for certain i, j. Whereas the $P_{i,j} = F_i + N_{i,j}$ split over the translation generators, the $\widetilde{P}_{i,j} = \widetilde{F}_i + N_{i,j}$ do not. Although each $\widetilde{F}_i$ is isomorphic to its counterpart F_i as a Lie algebra, it is *non-conjugate* to F_i within the Poincaré Lie algebra, as its generators are irrevocably "tied up" in linear combinations with the translation generators.

In this thesis we work in the context of all admissible symmetry reductions of the Maxwell-Dirac system, but we illustrate the method with a small selection of test cases. The standard limits of spherical and cylindrical symmetry (subgroups $P_{3,4}$ and $P_{12,8}$ in the PWZ list) exemplify subgroups arising from three-dimensional geometry, biased towards a particular reference frame. A subgroup not explicitly covered here, but an interesting extension of the spherical case is that of the hyperbolic symmetry subgroup $SO(2,1)$, represented in the PWZ list by $P_{4,4}$. This illustrates the case of a simple, but non-compact Lie algebra, and would make an interesting comparison case to more involved analyses of the $SO(3)$ Maxwell-Dirac symmetry reduction, due to its algebraic similarity. Finally, we take up two cases (with solvable Lie algebras) which specify one-parameter families of symmetries. The first, $P_{11,2}$, features an unusual "screw" generator, which is a parametric combination of a translation and a rotation about the corresponding axis. The second, $\widetilde{P}_{13,10}$ is a non-splitting subalgebra, with a parameter fixing the amount of translation generator entrained in the definition of a certain Lorentz generator in a minimal presentation.

In this chapter, we will calculate the scalar and vector field forms invariant under each subgroup, covering the actual Maxwell-Dirac reductions in the next chapter. As mentioned, the spherical and cylindrical cases reduce to a complicated system of nonlinear PDEs, there is no solution for $P_{11,2}$, and for $P_{13,10}$ the Maxwell-Dirac system reduces to a set of algebraic relations. Initially, we use the method described by Olver [34] to obtain a reduced set of independent variables, "invariants" henceforth, jointly invariant under the action of all the generators of the subgroup. It follows that arbitrary functions of the invariants will also be invariant under subgroup transformations, that is, they constitute solutions to the PDEs corresponding to a symmetric infinitesimal group action. Components of the invariant four-vector field must also be solutions to the PDEs corresponding to invariance under the group action. Such a set of differential equations is provided by the Lie derivative [44], which defines the directional derivative of a tensor field of rank (k,l) along the infinitesimal transformation vector field $\boldsymbol{\xi} \equiv \xi^\sigma \partial_\sigma$:

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\xi}} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} &= \xi^\sigma \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} - (\partial_\sigma \xi^{\mu_1}) T^{\sigma\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &- (\partial_\sigma \xi^{\mu_2}) T^{\mu_1\sigma\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} - \;\ldots\; + (\partial_{\nu_1}\xi^\sigma) T^{\mu_1\mu_2\ldots\mu_k}{}_{\sigma\nu_2\ldots\nu_l} \\ &+ (\partial_{\nu_2}\xi^\sigma) T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\sigma\ldots\nu_l} + \;\ldots,\end{aligned} \tag{3.1}$$

with invariance under $\boldsymbol{\xi}$ imposed by setting

$$\mathcal{L}_{\boldsymbol{\xi}} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = 0. \tag{3.2}$$

A scalar field ϕ is a rank $(0,0)$ tensor field, and the vector field with upper index Φ^μ is of rank $(1,0)$. Note that "rank" (k,l) in this context refers to the transformation properties as Lorentz tensor type objects, where k is the number of contravariant (upstairs) indices, and l is the number of covariant (downstairs) indices. Therefore, a tensor field such as Φ^μ is of rank $(1,0)$, and transforms as a contravariant vector field under Lorentz transformations. The field strength tensor $F_{\mu\nu}$ is of rank $(0,2)$, transforming as a tensor with two covariant Lorentz indices. We do not need to consider tensor fields with $k + l > 1$ however, since we have shown that $F_{\mu\nu}$ can be described in terms of scalar and four-vector fields of rank $(0, 0)$ and $(0, 1)$ respectively. Scalar and vector fields invariant under the transformation vector field $\boldsymbol{\xi}$ must solve the respective PDEs

$$\mathcal{L}_{\boldsymbol{\xi}}\phi = \xi^\sigma \partial_\sigma \phi = 0 \tag{3.3}$$

$$\mathcal{L}_{\boldsymbol{\xi}}\Phi^\mu = \xi^\sigma \partial_\sigma \Phi^\mu - (\partial_\sigma \xi^\mu)\Phi^\sigma = 0. \tag{3.4}$$

Solutions to (3.3) are calculated by using the method of characteristics to obtain the characteristic trajectories, which are the group invariants. Arbitrary scalar functions of these invariants solve (3.3), as can be confirmed via substitution. Solutions to (3.4) are obtained along the same lines, but the situation is complicated by the second term, which mixes some of the vector components. We can obtain a characteristic system of ODEs involving the Φ^μ components, from which we get algebraic expressions that allow us to make an accurate guess as to what forms components should take for invariance. The guessed solutions are confirmed by substituting into the equations generated by (3.4).

3.1 The Poincaré generators

One of the most important groups in special relativity is the Poincaré group $\mathcal{P}$, which consists of the Lorentz group of rotations and boosts, $SO(1, 3)$, as well as the Abelian group of translations in four dimensions, $T(4)$. Since $\mathcal{P}$ is a Lie group, we can take infinitesimal translations and rotations, building finite transformations from infinitesimal ones through exponentiation. This allows us to work with the mathematically simpler Lie algebra of the Poincaré group, $L(\mathcal{P})$, which forms a vector space with the generators as the basis. Subalgebras of $L(\mathcal{P})$ are described in terms of their constituent generators, and are by definition closed under a Lie bracket operation. The six infinitesimal generators of the Lorentz group are defined by

$$(l_{\alpha\beta})^\mu{}_\nu = \delta_\alpha{}^\mu \eta_{\beta\nu} - \delta_\beta{}^\mu \eta_{\alpha\nu} \qquad (\alpha, \beta = 0, 1, 2, 3), \tag{3.5}$$

with $l_{\alpha\beta} = -l_{\beta\alpha}$, and an arbitrary infinitesimal Lorentz transformation on the coordinate frame is

$$\Lambda^\mu{}_\nu x^\nu = [I + (1/2)\omega^{\alpha\beta} l_{\alpha\beta}]^\mu{}_\nu x^\nu, \tag{3.6}$$

where $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ are the six infinitesimal parameters associated with each generator. In explicit matrix form, the Lorentz generators are

$$l_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; l_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; l_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$l_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; l_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; l_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.7}$$

The first three generators are boosts along the x, y and z-axes respectively and the bottom three generators correspond to rotations in the $x-y$, $x-z$ and $y-z$ planes. Is is easy to show using (3.5) that the Lie bracket of the Lorentz algebra is

$$[l_{\alpha\beta}, l_{\gamma\delta}] = \eta_{\alpha\delta} l_{\beta\gamma} + \eta_{\beta\gamma} l_{\alpha\delta} - \eta_{\alpha\gamma} l_{\beta\delta} - \eta_{\beta\delta} l_{\alpha\gamma}. \tag{3.8}$$

A commonly used representation of these generators is

$$K_1 = -l_{01}, \quad K_2 = -l_{02}, \quad K_3 = -l_{03}, \quad L_1 = l_{23}, \quad L_2 = -l_{13}, \quad L_3 = l_{12}. \tag{3.9}$$

We define the components of the transformation vector field $\boldsymbol{\xi}_X$ corresponding to generator X, to be ${\xi_X}^\mu \equiv X^\mu{}_\nu x^\nu$. In the $K - L$ representation, these vector fields are

$$\begin{aligned}
{\xi_{K_1}}^\mu &= \begin{pmatrix} x \\ t \\ 0 \\ 0 \end{pmatrix}, \; {\xi_{K_2}}^\mu = \begin{pmatrix} y \\ 0 \\ t \\ 0 \end{pmatrix}, \; {\xi_{K_3}}^\mu = \begin{pmatrix} z \\ 0 \\ 0 \\ t \end{pmatrix}, \\
{\xi_{L_1}}^\mu &= \begin{pmatrix} 0 \\ 0 \\ -z \\ y \end{pmatrix}, \; {\xi_{L_2}}^\mu = \begin{pmatrix} 0 \\ z \\ 0 \\ -x \end{pmatrix}, \; {\xi_{L_3}}^\mu = \begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}
\end{aligned} \tag{3.10}$$

The vector fields ${\xi_{P_\nu}}^\mu$ corresponding to the infinitesimal translation operators P_ν are simply four vectors with components δ^μ_ν that act on the coordinate space additively

$$P_\nu \cdot x: \; x^\mu \to x^\mu + \varepsilon {\xi_{P_\nu}}^\mu = x^\mu + \varepsilon \delta^\mu_\nu. \tag{3.11}$$

The generators of all of the Poincaré subalgebras are listed by PWZ [35] using an alternative representation to that defined above, but which allows convenient extension to larger subgroups of the conformal group of spacetime transformations, such as the similitude (Weyl) group [35], [36]. It is important to note that if any extensions of the Maxwell-Dirac symmetry reduction to the conformal group are undertaken, the physical system must be restricted to *massless* particles only. The PWZ Lorentz generators are

$$\begin{aligned}
&B_1 = 2L_3 = 2l_{12}, \quad B_2 = -2K_3 = 2l_{03}, \quad B_3 = -L_2 - K_1 = l_{13} + l_{01}, \\
&B_4 = L_1 - K_2 = l_{23} + l_{02}, \quad B_5 = L_2 - K_1 = -l_{13} + l_{01}, \\
&B_6 = L_1 + K_2 = l_{23} - l_{02},
\end{aligned} \tag{3.12}$$

with explicit matrix form

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
B_4 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; B_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; B_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{3.13}$$

In this representation, it is appropriate to replace t and z with the light cone coordinates $l_+ \equiv t+z$ and $l_- \equiv t-z$. The transformation vector fields for each generator are

$$\xi_{B_1}{}^{\mu} = \begin{pmatrix} 0 \\ -2y \\ 2x \\ 0 \end{pmatrix}, \ \xi_{B_2}{}^{\mu} = \begin{pmatrix} -l_+ + l_- \\ 0 \\ 0 \\ -l_+ - l_- \end{pmatrix}, \ \xi_{B_3}{}^{\mu} = \begin{pmatrix} -x \\ -l_+ \\ 0 \\ x \end{pmatrix},$$

$$\xi_{B_4}{}^{\mu} = \begin{pmatrix} -y \\ 0 \\ -l_+ \\ y \end{pmatrix}, \ \xi_{B_5}{}^{\mu} = \begin{pmatrix} -x \\ -l_- \\ 0 \\ -x \end{pmatrix}, \ \xi_{B_6}{}^{\mu} = \begin{pmatrix} y \\ 0 \\ l_- \\ y \end{pmatrix}. \tag{3.14}$$

The translation generators in the PWZ representation are

$$X_1 = (1/2)(P_0 - P_3), \quad X_2 = P_2, \quad X_3 = -P_1, \quad X_4 = (1/2)(P_0 + P_3), \tag{3.15}$$

where X_1 and X_4 correspond to translations along the l_- and l_+ axes respectively. The corresponding (constant) vector fields are

$$\xi_{X_1}{}^{\mu} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ -1/2 \end{pmatrix}, \ \xi_{X_2}{}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \xi_{X_3}{}^{\mu} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \ \xi_{X_4}{}^{\mu} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}. \tag{3.16}$$

There is another B-generator, a composite of B_1 and B_2, which is present in the PWZ F_5 and F_{11} subalgebras of the Lorentz group, as well as the $P_{5,i}$ and $P_{11,i}$ subalgebras of the Poincaré group

$$B_\varphi = \cos\varphi B_1 + \sin\varphi B_2, \quad 0 < \varphi < \pi, \quad \varphi \neq \pi/2. \tag{3.17}$$

This generator corresponds to a simultaneous rotation around, and boost along the z-axis, the so-called "screw" group, $S(1)$. The continuous parameter varies the generator from being almost a pure rotation ($\varphi \approx 0$), to an almost pure boost ($\varphi \approx \pi/2$). The extreme cases when $\varphi \to 0^+$ and $\varphi \to \pi/2^-$, meaning $B_\varphi \to B_1$ and $B_\varphi \to B_2$, are actually Lorentz subalgebras in their own right, and are given the PWZ labels F_{12} and F_{13} respectively. The explicit matrix form and vector field for B_φ are

$$B_\varphi = \begin{pmatrix} 0 & 0 & 0 & -2\sin\varphi \\ 0 & 0 & -2\cos\varphi & 0 \\ 0 & 2\cos\varphi & 0 & 0 \\ -2\sin\varphi & 0 & 0 & 0 \end{pmatrix}, \quad \xi_{B_\varphi}{}^{\mu} = \begin{pmatrix} -2z\sin\varphi \\ -2y\cos\varphi \\ 2x\cos\varphi \\ -2t\sin\varphi \end{pmatrix}. \tag{3.18}$$

3.2 Spherical symmetry (subgroup $P_{3,4}$)

The condition for scalar and vector fields to be spherically symmetric is that they be invariant under the action of the $SO(3)$ group, which consists of the three rotation generators L_1, L_2 and L_3. The Lie derivative of a scalar field invariant under L_1 is

$$\mathcal{L}_{L_1}\phi = \xi_{L_1}{}^{\sigma}\partial_\sigma\phi = -z\partial_y\phi + y\partial_z\phi = 0, \tag{3.19}$$

which yields the characteristic system

$$\frac{\mathrm{d}y}{-z} = \frac{\mathrm{d}z}{y}, \tag{3.20}$$

giving us the L_1 invariant $\rho = \sqrt{y^2 + z^2}$. The requirement for invariance under L_2 is

$$\mathcal{L}_{L_2}\phi = \xi_{L_2}{}^{\sigma}\partial_\sigma\phi = z\partial_x\phi - x\partial_z\phi = 0. \tag{3.21}$$

To impose that ϕ to be jointly invariant under L_1 and L_2, we require that the solution to (3.21) be a function of t, x and ρ. Using the chain rule, we find that the PDE becomes independent from z explicitly

$$\rho\partial_x\phi - x\partial_\rho\phi = 0. \tag{3.22}$$

The characteristic equation is

$$\frac{\mathrm{d}x}{\rho} = \frac{\mathrm{d}\rho}{-x}, \tag{3.23}$$

from which we obtain the joint invariant $r = \sqrt{x^2 + y^2 + z^2}$. Lastly, we require that

$$\mathcal{L}_{L_3}\phi = \xi_{L_3}{}^{\sigma}\partial_\sigma\phi = -y\partial_x\phi + x\partial_y\phi = 0, \tag{3.24}$$

which is automatically satisfied by $\phi(t, r)$. Now consider the four vector field Φ^μ invariant under L_1. From (3.4) for $\mu = 0 - 3$, we obtain the following PDEs

$$\begin{aligned}
\mathcal{L}_{L_1}\Phi^0 &= -z\partial_y\Phi^0 + y\partial_z\Phi^0 = 0, && (3.25a)\\
\mathcal{L}_{L_1}\Phi^1 &= -z\partial_y\Phi^1 + y\partial_z\Phi^1 = 0, && (3.25b)\\
\mathcal{L}_{L_1}\Phi^2 &= -z\partial_y\Phi^2 + y\partial_z\Phi^2 + \Phi^3 = 0, && (3.25c)\\
\mathcal{L}_{L_1}\Phi^3 &= -z\partial_y\Phi^3 + y\partial_z\Phi^3 - \Phi^2 = 0. && (3.25d)
\end{aligned}$$

Likewise, for L_2 we obtain

$$\begin{aligned}
\mathcal{L}_{L_2}\Phi^0 &= z\partial_x\Phi^0 - x\partial_z\Phi^0 = 0, && (3.26a)\\
\mathcal{L}_{L_2}\Phi^1 &= z\partial_x\Phi^1 - x\partial_z\Phi^1 - \Phi^3 = 0, && (3.26b)\\
\mathcal{L}_{L_2}\Phi^2 &= z\partial_x\Phi^2 - x\partial_z\Phi^2 = 0, && (3.26c)\\
\mathcal{L}_{L_2}\Phi^3 &= z\partial_x\Phi^3 - x\partial_z\Phi^3 + \Phi^1 = 0, && (3.26d)
\end{aligned}$$

and for L_3 we get

$$\begin{aligned}
\mathcal{L}_{L_3}\Phi^0 &= -y\partial_x\Phi^0 + x\partial_y\Phi^0 = 0, && (3.27a)\\
\mathcal{L}_{L_3}\Phi^1 &= -y\partial_x\Phi^1 + x\partial_y\Phi^1 + \Phi^2 = 0, && (3.27b)\\
\mathcal{L}_{L_3}\Phi^2 &= -y\partial_x\Phi^2 + x\partial_y\Phi^2 - \Phi^1 = 0, && (3.27c)\\
\mathcal{L}_{L_3}\Phi^3 &= -y\partial_x\Phi^3 + x\partial_y\Phi^3 = 0. && (3.27d)
\end{aligned}$$

Noticing that Φ^0 obeys the same set of PDEs as ϕ, we can immediately conclude that $\Phi^0 = a(t, r)$. By taking the combination $x\mathcal{L}_{L_1}\Phi^i + y\mathcal{L}_{L_2}\Phi^i + z\mathcal{L}_{L_3}\Phi^i$ for $i = 1, 2, 3$, we simplify the other PDEs to the algebraic set

$$z\Phi^2 - y\Phi^3 = 0, \tag{3.28a}$$

$$x\Phi^3 - z\Phi^1 = 0, \tag{3.28b}$$

$$y\Phi^1 - x\Phi^2 = 0, \tag{3.28c}$$

giving us the solution $\Phi^i = x^i b(t,r)$. So $SO(3)$ invariant four vector fields must have the general form

$$\Phi^\mu = \begin{pmatrix} a(t,r) \\ xb(t,r) \\ yb(t,r) \\ zb(t,r) \end{pmatrix}. \tag{3.29}$$

3.3 Cylindrical symmetry (subgroup $P_{12,8}$)

For tensor fields to be cylindrically symmetric, they must be invariant under rotation around, and translation along, a single axis. Choosing the rotation plane to be $x-y$, the axis must be z, so the infinitesimal invariance generators are L_3 and P_3. The scalar field must satisfy the relatively trivial PDEs

$$\mathcal{L}_{P_3}\phi = \xi_{P_3}{}^\sigma \partial_\sigma \phi = \partial_z \phi = 0, \tag{3.30}$$

$$\mathcal{L}_{L_3}\phi = \xi_{L_3}{}^\sigma \partial_\sigma \phi = -y\partial_x \phi + x\partial_y \phi = 0. \tag{3.31}$$

The first equation tells us that ϕ is independent of z, and from the second we obtain the invariant $\rho = \sqrt{x^2+y^2}$. Cylindrical symmetry requires that scalar fields be functions of t and ρ only. The vector fields must satisfy

$$\mathcal{L}_{P_3}\Phi^\mu = \xi_{P_3}{}^\sigma \partial_\sigma \Phi^\mu = 0, \tag{3.32}$$

as well as equations (3.27a) to (3.27d). We can immediately conclude that $\Phi^0 = a(t,\rho)$ and $\Phi^3 = d(t,\rho)$, since they solve the same equations as ϕ. We can also say that Φ^1 and Φ^2 are independent of z, but due to components mixing, they are not pure functions of t and ρ. From (3.27b) and (3.27c), we obtain the respective characteristic systems

$$\frac{\mathrm{d}x}{-y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}\Phi^1}{-\Phi^2}, \tag{3.33a}$$

$$\frac{\mathrm{d}x}{-y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}\Phi^2}{\Phi^1}, \tag{3.33b}$$

obviously implying that

$$\frac{\mathrm{d}\Phi^1}{-\Phi^2} = \frac{\mathrm{d}\Phi^2}{\Phi^1}. \tag{3.34}$$

Integrating and taking into account the fact that arbitrary functions of t and ρ are constant along characteristic curves, we obtain the algebraic constraint

$$(\Phi^1)^2 + (\Phi^2)^2 = f(t,\rho), \tag{3.35}$$

which accepts solutions of the form $\Phi^1 = xb(t,\rho) - yc(t,\rho)$ and $\Phi^2 = yb(t,\rho) + xc(t,\rho)$. This gives us the form of the cylindrically symmetric four vector field

$$\Phi^\mu = \begin{pmatrix} a(t,\rho) \\ xb(t,\rho) - yc(t,\rho) \\ yb(t,\rho) + xc(t,\rho) \\ d(t,\rho) \end{pmatrix}. \tag{3.36}$$

3.4 $P_{11,2}$ symmetry ("screw" subgroup)

The Poincaré subalgebra $P_{11,2}$ as defined by PWZ consists of the single Lorentz generator B_φ, and the three translation generators X_1, X_2 and X_3. In this section, we will find the symmetric form of the fields for the B_φ generator first, then proceed through the translation generators in numerical order. The process is a cumulative one, in that once we have derived the form of the B_φ invariant fields, we apply the X_1 invariance condition to them in this form, resulting in a more restricted form, and so on.

3.4.1 B_φ invariant fields

From (3.18), we can see that the B_φ invariance condition for a scalar field is

$$\mathcal{L}_{B_\varphi}\phi = {\xi_{B_\varphi}}^\sigma \partial_\sigma \phi = -2z\sin\varphi\,\partial_t\phi - 2y\cos\varphi\,\partial_x\phi + 2x\cos\varphi\,\partial_y\phi - 2t\sin\varphi\,\partial_z\phi = 0. \tag{3.37}$$

Since we are using the light cone coordinates, we must use the chain rule to rewrite the derivatives

$$\partial_t\phi = \partial_+\phi + \partial_-\phi, \tag{3.38}$$
$$\partial_z\phi = \partial_+\phi - \partial_-\phi, \tag{3.39}$$

resulting in the PDE

$$-l_+\sin\varphi\,\partial_+\phi + l_-\sin\varphi\,\partial_-\phi - y\cos\varphi\,\partial_x\phi + x\cos\varphi\,\partial_y\phi = 0, \tag{3.40}$$

where for simplicity, we have defined $\partial_+ \equiv \partial/\partial l_+$ and $\partial_- \equiv \partial/\partial l_-$. From the method of characteristics, we get the system of six ODEs

$$\frac{\mathrm{d}l_+}{-l_+\sin\varphi} = \frac{\mathrm{d}l_-}{l_-\sin\varphi} = \frac{\mathrm{d}x}{-y\cos\varphi} = \frac{\mathrm{d}y}{x\cos\varphi}, \tag{3.41}$$

from which we obtain the six invariants

$$|L| = |l_+l_-| = |t^2 - z^2|, \tag{3.42a}$$
$$\rho = \sqrt{x^2+y^2}, \tag{3.42b}$$
$$\alpha = \cos\varphi\ \ln|l_+| - \sin\varphi\ \arcsin(x/\rho), \tag{3.42c}$$
$$\beta = \cos\varphi\ \ln|l_+| + \sin\varphi\ \arcsin(y/\rho), \tag{3.42d}$$
$$\gamma = \cos\varphi\ \ln|l_-| + \sin\varphi\ \arcsin(x/\rho), \tag{3.42e}$$
$$\delta = \cos\varphi\ \ln|l_-| - \sin\varphi\ \arcsin(y/\rho), \tag{3.42f}$$

where we obtain the last four invariants via integration by recognizing that ρ is a constant in the characteristic system. Not all of these invariants are independent. For example, adding α and γ, then rearranging gives

$$|L| = \exp\left(\frac{\alpha+\gamma}{\cos\varphi}\right). \tag{3.43}$$

In addition to $|L|$ and ρ, we can construct a neat form for a third invariant from the list α, β, γ, δ. Consider the combination

$$\alpha - \delta = \beta - \gamma = \cos\varphi\,(\ln|l_+| - \ln|l_-|) + \sin\varphi\,[\arcsin(y/\rho) - \arcsin(x/\rho)]. \quad (3.44)$$

After some manipulation, we find that

$$\exp[(\alpha - \delta)/\cos\varphi] = (l_+/l_-)\exp\{\tau\arctan[(y^2 - x^2)/2xy]\}, \quad (3.45)$$

where $\tau \equiv \tan\varphi$. Using the logarithmic form of arctan, and introducing polar coordinates in the $x - y$ plane

$$y + \mathrm{i}x = \rho\mathrm{e}^{\mathrm{i}\chi}, \quad (3.46)$$

where $\chi \equiv \arctan(x/y)$, we find that

$$\exp[(\alpha - \delta)/\cos\varphi] = -(l_+/l_-)\mathrm{e}^{-2\tau\chi}. \quad (3.47)$$

Taking the negative reciprocal of this, we arrive at the form for the new invariant

$$\zeta_\varphi = (l_-/l_+)\mathrm{e}^{2\tau\chi}, \quad (3.48)$$

with the φ subscript indicating that the invariant is dependent on the value of the free group parameter. In summary, by imposing B_φ invariance, we have gone from the independent variable set (l_+, l_-, x, y) to the reduced set $(|L|, \rho, \zeta_\varphi)$. Arbitrary scalar functions of the latter set are solutions to (3.40), which can be checked via substituting the partial derivatives of $\phi(|L|, \rho, \zeta_\varphi)$

$$\partial_+\phi = l_-(L/|L|)\partial_{|L|}\phi - (\zeta_\varphi/l_+)\partial_\zeta\phi, \quad (3.49a)$$
$$\partial_-\phi = l_+(L/|L|)\partial_{|L|}\phi + (\zeta_\varphi/l_-)\partial_\zeta\phi, \quad (3.49b)$$
$$\partial_x\phi = (x/\rho)\partial_\rho\phi + (2\tau\zeta_\varphi y/\rho^2)\partial_\zeta\phi, \quad (3.49c)$$
$$\partial_y\phi = (y/\rho)\partial_\rho\phi - (2\tau\zeta_\varphi x/\rho^2)\partial_\zeta\phi. \quad (3.49d)$$

Now the vector field components must satisfy

$$\mathcal{L}_{B_\varphi}\Phi^\mu = \xi_{B_\varphi}{}^\sigma\partial_\sigma\Phi^\mu - (\partial_\sigma\xi_{B_\varphi}{}^\mu)\Phi^\sigma = 0, \quad (3.50)$$

which for $\mu = 0 - 3$ gives us

$$-\,l_+\sin\varphi\;\partial_+\Phi^0 + l_-\sin\varphi\;\partial_-\Phi^0 - y\cos\varphi\;\partial_x\Phi^0 + x\cos\varphi\;\partial_y\Phi^0 + \sin\varphi\;\Phi^3 = 0, \quad (3.51a)$$
$$-\,l_+\sin\varphi\;\partial_+\Phi^1 + l_-\sin\varphi\;\partial_-\Phi^1 - y\cos\varphi\;\partial_x\Phi^1 + x\cos\varphi\;\partial_y\Phi^1 + \cos\varphi\;\Phi^2 = 0, \quad (3.51b)$$
$$-\,l_+\sin\varphi\;\partial_+\Phi^2 + l_-\sin\varphi\;\partial_-\Phi^2 - y\cos\varphi\;\partial_x\Phi^2 + x\cos\varphi\;\partial_y\Phi^2 - \cos\varphi\;\Phi^1 = 0, \quad (3.51c)$$
$$-\,l_+\sin\varphi\;\partial_+\Phi^3 + l_-\sin\varphi\;\partial_-\Phi^3 - y\cos\varphi\;\partial_x\Phi^3 + x\cos\varphi\;\partial_y\Phi^3 + \sin\varphi\;\Phi^0 = 0. \quad (3.51d)$$

The method of characteristics gives us a system of ODEs like (3.41) for each equation, but with an extra $\mathrm{d}\Phi^\mu$ part. Equating these parts gives us an ODE system involving only the vector field components

$$\frac{\mathrm{d}\Phi^0}{-\sin\varphi\ \Phi^3} = \frac{\mathrm{d}\Phi^1}{-\cos\varphi\ \Phi^2} = \frac{\mathrm{d}\Phi^2}{\cos\varphi\ \Phi^1} = \frac{\mathrm{d}\Phi^3}{-\sin\varphi\ \Phi^0}. \tag{3.52}$$

Integrating this ODE system, we obtain a list of algebraic constraints on the vector field components

$$A^2(|L|,\rho,\zeta_\varphi) = (\Phi^0)^2 - (\Phi^3)^2, \tag{3.53a}$$
$$B^2(|L|,\rho,\zeta_\varphi) = (\Phi^1)^2 + (\Phi^2)^2, \tag{3.53b}$$
$$f(|L|,\rho,\zeta_\varphi) = \tau \arcsin(\Phi^1/B) - \operatorname{arccosh}(\Phi^0/A), \tag{3.53c}$$
$$g(|L|,\rho,\zeta_\varphi) = \tau \arcsin(\Phi^2/B) + \operatorname{arccosh}(\Phi^0/A), \tag{3.53d}$$
$$h(|L|,\rho,\zeta_\varphi) = \tau \arcsin(\Phi^1/B) - \operatorname{arcsinh}(\Phi^3/A), \tag{3.53e}$$
$$k(|L|,\rho,\zeta_\varphi) = \tau \arcsin(\Phi^2/B) + \operatorname{arcsinh}(\Phi^3/A), \tag{3.53f}$$

where we have used the fact that arbitrary functions of the invariants are constants in the characteristic ODE system. By introducing angular coordinates in the $\Phi^0-\Phi^3$ and $\Phi^1-\Phi^2$ planes, we find that $f = h$ and $g = k$, so we effectively have only four independent constraints. After some investigation, we find that the f and g angular constraints provide a superfluous level of detail that can be eliminated by a simple relabeling of functions. Focusing on the quadratic constraints (3.53a) and (3.53b), we find that an appropriate generic form for the four vector field is

$$\Phi^\mu = \begin{pmatrix} ta(|L|,\rho,\zeta_\varphi) + zb(|L|,\rho,\zeta_\varphi) \\ xc(|L|,\rho,\zeta_\varphi) - yd(|L|,\rho,\zeta_\varphi) \\ yc(|L|,\rho,\zeta_\varphi) + xd(|L|,\rho,\zeta_\varphi) \\ za(|L|,\rho,\zeta_\varphi) + tb(|L|,\rho,\zeta_\varphi) \end{pmatrix}. \tag{3.54}$$

Verification can be obtained by substituting into (3.51a)-(3.51d), (3.53a) and (3.53b). Note that since the Poincaré subgroup $P_{11,6}$ consists of the single Lorentz generator B_φ, this is the invariant four vector field form for that subgroup. $P_{11,2}$ consists of B_φ, as well as the three translation generators X_1, X_2 and X_3, the corresponding vector fields being listed in (3.16).

3.4.2 B_φ, X_1 invariant fields

The condition that scalar fields are simultaneously invariant under B_φ and X_1 is

$${\xi_{X_1}}^\sigma \partial_\sigma \phi(|L|,\rho,\zeta_\varphi) = \partial_- \phi(|L|,\rho,\zeta_\varphi) = 0, \tag{3.55}$$

which upon applying the chain rule and multiplying through by l_-, gives the PDE

$$|L|\partial_{|L|}\phi + \zeta_\varphi \partial_\zeta \phi = 0. \tag{3.56}$$

The corresponding characteristic ODE for this equation is

$$\frac{\mathrm{d}|L|}{|L|} = \frac{\mathrm{d}\zeta_\varphi}{\zeta_\varphi}, \tag{3.57}$$

which can be integrated to yield the $\{B_\varphi, X_1\}$ joint invariant

$$\widetilde{\zeta}_\varphi \equiv L/\zeta_\varphi = l_+^2 \mathrm{e}^{-2\tau\chi}. \tag{3.58}$$

So in order to be jointly invariant along the characteristics generated by both B_φ and X_1, the scalar field ϕ must be a function of $(\rho, \widetilde{\zeta}_\varphi)$ only. Since neither ρ, nor $\widetilde{\zeta}_\varphi$ are dependent on l_-, it is obvious that $\phi(\rho, \widetilde{\zeta}_\varphi)$ solves (3.55).

For four vector fields to be simultaneously invariant under B_φ and X_1, we require that

$$\mathcal{L}_{X_1}\Phi^\mu = \xi_{X_1}{}^\sigma \partial_\sigma \Phi^\mu - (\partial_\sigma \xi_{X_1}{}^\mu)\Phi^\sigma = \partial_- \Phi^\mu = 0, \tag{3.59}$$

where Φ^μ are the components of (3.54). The second term is zero because $\xi_{X_1}{}^\mu$ is a constant vector field. Carrying out the derivatives yields the four PDEs

$$l_-(a-b)/2 + t(|L|\partial_{|L|}a + \zeta_\varphi \partial_\zeta a) + z(|L|\partial_{|L|}b + \zeta_\varphi \partial_\zeta b) = 0, \tag{3.60a}$$
$$x(|L|\partial_{|L|}c + \zeta_\varphi \partial_\zeta c) - y(|L|\partial_{|L|}d + \zeta_\varphi \partial_\zeta d) = 0, \tag{3.60b}$$
$$y(|L|\partial_{|L|}c + \zeta_\varphi \partial_\zeta c) + x(|L|\partial_{|L|}d + \zeta_\varphi \partial_\zeta d) = 0, \tag{3.60c}$$
$$-\,l_-(a-b)/2 + z(|L|\partial_{|L|}a + \zeta_\varphi \partial_\zeta a) + t(|L|\partial_{|L|}b + \zeta_\varphi \partial_\zeta b) = 0, \tag{3.60d}$$

where we have taken the additional step of multiplying through by l_-. Adding (3.60a) to (3.60d) and subtracting (3.60d) from (3.60a) gives the two respective equations

$$|L|\partial_{|L|}a + \zeta_\varphi \partial_\zeta a = -(|L|\partial_{|L|}b + \zeta_\varphi \partial_\zeta b), \tag{3.61a}$$
$$b - a = |L|\partial_{|L|}a + \zeta_\varphi \partial_\zeta a - (|L|\partial_{|L|}b + \zeta_\varphi \partial_\zeta b). \tag{3.61b}$$

The first equation can be substituted into the second to eliminate either the a or b derivatives from the right hand side, giving us non-homogeneous PDEs for a and b:

$$(b-a)/2 = |L|\partial_{|L|}a + \zeta_\varphi \partial_\zeta a, \tag{3.62a}$$
$$(a-b)/2 = |L|\partial_{|L|}b + \zeta_\varphi \partial_\zeta b, \tag{3.62b}$$

to which we can apply the method of characteristics, giving us the respective ODE systems

$$\frac{\mathrm{d}|L|}{|L|} = \frac{\mathrm{d}\zeta_\varphi}{\zeta_\varphi} = \frac{2\mathrm{d}a}{b-a}, \tag{3.63a}$$
$$\frac{\mathrm{d}|L|}{|L|} = \frac{\mathrm{d}\zeta_\varphi}{\zeta_\varphi} = \frac{2\mathrm{d}b}{a-b}. \tag{3.63b}$$

The first two parts of each ODE system again tell us that characteristics lie along curves of constant $\widetilde{\zeta}_\varphi = l_+^2 \mathrm{e}^{-2\tau\chi}$. We can equate the right hand sides of both ODE systems and integrate to yield the algebraic constraint on the forms of a and b

$$a(|L|, \rho, \zeta_\varphi) + b(|L|, \rho, \zeta_\varphi) = f(\rho, \widetilde{\zeta}_\varphi), \tag{3.64}$$

where f is a constant in this ODE system. This constraint is satisfied if we simply impose that a and b now be functions of ρ and $\widetilde{\zeta}_\varphi$. Substituting these into the $\mu = 0$ Lie derivative gives

$$\partial_- \Phi^0 = \partial_-[ta(\rho, \widetilde{\zeta}_\varphi) + zb(\rho, \widetilde{\zeta}_\varphi)] = a/2 - b/2 = 0, \tag{3.65}$$

which is satisfied only if $a = b$. The $\mu = 3$ equation gives the same result, so we conclude that the form of the Φ^0 and Φ^3 components is

$$\Phi^0 = \Phi^3 = l_+ a(\rho, \widetilde{\zeta}_\varphi). \tag{3.66}$$

Moving on to the $\mu = 1, 2$ PDEs, taking the combinations x(3.60b)+y(3.60c) as well as x(3.60c) $-y$(3.60b) yields the two respective PDEs

$$|L|\partial_{|L|} c + \zeta_\varphi \partial_\zeta c = 0, \tag{3.67a}$$
$$|L|\partial_{|L|} d + \zeta_\varphi \partial_\zeta d = 0. \tag{3.67b}$$

Since these equations are of exactly the same form as the scalar field PDE (3.56), we conclude that c and d must be functions of $(\rho, \widetilde{\zeta}_\varphi)$. Our $\{B_\varphi, X_1\}$ invariant four vector field is therefore of the form

$$\Phi^\mu = \begin{pmatrix} l_+ a(\rho, \widetilde{\zeta}_\varphi) \\ xb(\rho, \widetilde{\zeta}_\varphi) - yc(\rho, \widetilde{\zeta}_\varphi) \\ yb(\rho, \widetilde{\zeta}_\varphi) + xc(\rho, \widetilde{\zeta}_\varphi) \\ l_+ a(\rho, \widetilde{\zeta}_\varphi) \end{pmatrix}. \tag{3.68}$$

Note that this invariant form corresponds to the $P_{11,5}$ Poincaré subalgebra.

3.4.3 B_φ, X_1, X_2 invariant fields

Scalar fields simultaneously invariant under B_φ, X_1 and X_2 must satisfy

$$\xi_{X_2}{}^\sigma \partial_\sigma \phi(\rho, \widetilde{\zeta}_\varphi) = \partial_y \phi(\rho, \widetilde{\zeta}_\varphi) = 0, \tag{3.69}$$

which when applying the chain rule, gives us the PDE

$$y(1/\rho)\partial_\rho \phi + x(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} \phi = 0. \tag{3.70}$$

For this equation to hold for all x, y, it must be that the derivative terms are zero, implying that ϕ is a constant.

For four vector fields invariant under these generators, we require

$$\xi_{X_2}{}^\sigma \partial_\sigma \Phi^\mu - (\partial_\sigma \xi_{X_2}{}^\mu)\Phi^\sigma = \partial_y \Phi^\mu = 0, \tag{3.71}$$

with Φ^μ components given by (3.68). The equation for the $\mu = 0$ and $\mu = 3$ components is

$$y(1/\rho)\partial_\rho a + x(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} a = 0, \tag{3.72}$$

which has exactly the same form as (3.70), so a is constant. After applying the chain rule, the $\mu = 1$ and $\mu = 2$ equations respectively are

$$c = x^2(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} b + xy(1/\rho)\partial_\rho b - xy(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} c - y^2(1/\rho)\partial_\rho c, \tag{3.73a}$$
$$b = -x^2(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} c - xy(1/\rho)\partial_\rho c - xy(2\tau\widetilde{\zeta}_\varphi/\rho^2)\partial_{\widetilde{\zeta}} b - y^2(1/\rho)\partial_\rho b. \tag{3.73b}$$

Since it is established that c and b are both functions of ρ and $\widetilde{\zeta}_\varphi$, for these expressions to hold for all x, y, it must be that $b = c = 0$. Intuitively speaking, if we imagine the form of a vector field symmetric under rotations around the $z-$axis, with x and y components Φ^1 and Φ^2, including the additional requirement of symmetry along $y-$axis translations forces both components to be zero. We have determined the form of the $\{B_\varphi, X_1, X_2\}$ invariant four vector field to be

$$\Phi^\mu = \begin{pmatrix} l_+ a \\ 0 \\ 0 \\ l_+ a \end{pmatrix}, \tag{3.74}$$

where a is a constant.

3.4.4 B_φ, X_1, X_2, X_3 invariant fields

Finally, for full $P_{11,2}$ symmetry, we require that globally constant scalars satisfy

$$\xi_{X_3}{}^\sigma \partial_\sigma \phi = -\partial_x \phi = 0, \tag{3.75}$$

which they obviously do. Four vector fields must satisfy

$$\xi_{X_3}{}^\sigma \partial_\sigma \Phi^\mu - (\partial_\sigma \xi_{X_3}{}^\mu)\Phi^\sigma = -\partial_x \Phi^\mu = 0. \tag{3.76}$$

Since Φ^0 and Φ^3 in (3.74) are already independent of x, (3.76) is automatically satisfied. We therefore conclude that (3.74) is the full $P_{11,2}$ invariant four vector field form.

3.5 $\widetilde{P}_{13,10}$ symmetry

The final example we cover here is the non-splitting $\widetilde{P}_{13,10}$ Poincaré subalgebra, consisting of the generators $B_2 + \lambda X_2$ ($\lambda > 0$), X_1, X_3 and X_4. The non-splitting aspect is manifested in the fact that the pure Lorentz generator B_2 is "tied up" with the translation generator λX_2, where the non-zero parameter λ determines the relative weight of the translation part. To condense the notation, we define the abbreviated form of the non-split generator, $\widetilde{B}_\lambda \equiv B_2 + \lambda X_2$, where the tilde is to emphasize that it is not a pure Lorentz generator. As in the previous section, we will consider the Lorentz-translation generator $\widetilde{B}_\lambda$ first, then sequentially apply the translation generators in numerical order.

3.5.1 $\widetilde{B}_\lambda$ invariant fields

B_2 generates hyperbolic rotations in the $t - z$ plane, and X_2 generates translations along the $y-$axis. Since the individual vector fields point in orthogonal directions, the vector field corresponding to $\widetilde{B}_\lambda$ is simply the linear combination

$$\xi_{B_2}{}^\mu + \lambda \xi_{X_2}{}^\mu \equiv \xi_{\widetilde{B}_\lambda}{}^\mu = \begin{pmatrix} -l_+ + l_- \\ 0 \\ \lambda \\ -l_+ - l_- \end{pmatrix}. \tag{3.77}$$

A scalar field ϕ invariant under the action of $\widetilde{B}_\lambda$ solves

$$\mathcal{L}_{\widetilde{B}_\lambda}\phi = \xi_{\widetilde{B}_\lambda}{}^\sigma \partial_\sigma \phi = (-l_+ + l_-)\partial_t \phi + \lambda \partial_y \phi + (-l_+ - l_-)\partial_z \phi = 0. \tag{3.78}$$

Applying the chain rule (3.38), (3.39), this PDE becomes

$$-2l_+\partial_+\phi + 2l_-\partial_-\phi + \lambda\partial_y\phi = 0, \tag{3.79}$$

which has the set of characteristic ODEs

$$\frac{\mathrm{d}l_+}{-2l_+} = \frac{\mathrm{d}l_-}{2l_-} = \frac{\mathrm{d}y}{\lambda}. \tag{3.80}$$

Integrating these three equations yields the set of invariants

$$|L| = |l_+l_-| = |t^2 - z^2|, \tag{3.81a}$$
$$\alpha = |l_+|\mathrm{e}^{2y/\lambda}, \tag{3.81b}$$
$$\beta = |l_-|\mathrm{e}^{-2y/\lambda}, \tag{3.81c}$$

If α and β are invariants, then so is

$$\alpha\beta = |l_+||l_-|\mathrm{e}^{2y/\lambda}\mathrm{e}^{-2y/\lambda} = |L|, \tag{3.82}$$

implying that $|L|$ is not independent from the other two. Choosing ϕ to be an arbitrary function of α, β and x, we find that (3.79) is satisfied. For future reference, the partial derivatives of $\phi(\alpha, \beta, x)$ are

$$\partial_t\phi = \partial_\alpha\phi \cdot \partial_+\alpha + \partial_\beta\phi \cdot \partial_-\beta = (l_+/|l_+|)\mathrm{e}^{2y/\lambda}\partial_\alpha\phi + (l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta\phi, \tag{3.83a}$$
$$\partial_y\phi = \partial_\alpha\phi \cdot \partial_y\alpha + \partial_\beta\phi \cdot \partial_y\beta = (2|l_+|/\lambda)\mathrm{e}^{2y/\lambda}\partial_\alpha\phi - (2|l_-|/\lambda)\mathrm{e}^{-2y/\lambda}\partial_\beta\phi, \tag{3.83b}$$
$$\partial_z\phi = \partial_\alpha\phi \cdot \partial_+\alpha - \partial_\beta\phi \cdot \partial_-\beta = (l_+/|l_+|)\mathrm{e}^{2y/\lambda}\partial_\alpha\phi - (l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta\phi, \tag{3.83c}$$

where $\partial_t l_+ = \partial_z l_+ = \partial_t l_- = 1$ and $\partial_z l_- = -1$ are implicit. Invariant vector fields must solve

$$\mathcal{L}_{\widetilde{B}_\lambda}\Phi^\mu = \xi_{\widetilde{B}_\lambda}{}^\sigma\partial_\sigma\Phi^\mu - (\partial_\sigma\xi_{\widetilde{B}_\lambda}{}^\mu)\Phi^\sigma = 0, \tag{3.84}$$

which for $\mu = 0 - 3$ gives us the set of PDEs

$$-2l_+\partial_+\Phi^0 + 2l_-\partial_-\Phi^0 + \lambda\partial_y\Phi^0 + 2\Phi^3 = 0, \tag{3.85a}$$
$$-2l_+\partial_+\Phi^1 + 2l_-\partial_-\Phi^1 + \lambda\partial_y\Phi^1 = 0, \tag{3.85b}$$
$$-2l_+\partial_+\Phi^2 + 2l_-\partial_-\Phi^2 + \lambda\partial_y\Phi^2 = 0, \tag{3.85c}$$
$$-2l_+\partial_+\Phi^3 + 2l_-\partial_-\Phi^3 + \lambda\partial_y\Phi^3 + 2\Phi^0 = 0. \tag{3.85d}$$

Since (3.85b) and (3.85c) are the same form as (3.79), they have the same characteristic solution as the scalar field: $\Phi^1 = c(\alpha, \beta, x)$ and $\Phi^2 = d(\alpha, \beta, x)$. Applying the method of characteristics to the non-homogeneous PDEs (3.85a) and (3.85d), we obtain two systems of ODEs exactly like (3.80), but with extra $\mathrm{d}\Phi^\mu$ parts. Equating these parts gives

$$\frac{\mathrm{d}\Phi^0}{2\Phi^3} = \frac{\mathrm{d}\Phi^3}{2\Phi^0}, \tag{3.86}$$

which when integrated yields the algebraic constraint

$$(\Phi^0)^2 - (\Phi^3)^2 = f(\alpha, \beta, x). \tag{3.87}$$

This has the solution

$$\Phi^0 = l_+ a(\alpha, \beta, x) + l_- b(\alpha, \beta, x), \tag{3.88a}$$

$$\Phi^3 = l_+ a(\alpha, \beta, x) - l_- b(\alpha, \beta, x), \tag{3.88b}$$

which can be checked via substitution. We have determined the form of the $\widetilde{B}_\lambda$ invariant four vector field to be

$$\Phi^\mu = \begin{pmatrix} l_+ a + l_- b \\ c \\ d \\ l_+ a - l_- b \end{pmatrix}, \tag{3.89}$$

where a, b, c and d are functions of α, β and x. This form can be verified by inserting the appropriate components into (3.85a)-(3.85d), as well as (3.87).

3.5.2 $\widetilde{B}_\lambda$, X_1 invariant fields

Scalar fields invariant under both $\widetilde{B}_\lambda$ and X_1 must solve

$$\mathcal{L}_{X_1}\phi(\alpha, \beta, x) = \xi_{X_1}{}^\sigma \partial_\sigma \phi(\alpha, \beta, x) = (1/2)(\partial_t \phi - \partial_z \phi) = 0. \tag{3.90}$$

Applying (3.83a) and (3.83c) gives us

$$(l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta \phi = 0, \tag{3.91}$$

but since $l_-/|l_-| = \pm 1$ ($l_- \neq 0$) and $\mathrm{e}^{-2y/\lambda}$ are positive definite, we require

$$\partial_\beta \phi = 0, \tag{3.92}$$

implying that ϕ is independent of β. For the four vector field to be invariant under $\widetilde{B}_\lambda$ and X_1, we must have

$$\mathcal{L}_{X_1}\Phi^\mu = \xi_{X_1}{}^\sigma \partial_\sigma \Phi^\mu - (\partial_\sigma \xi_{X_1}{}^\mu)\Phi^\sigma = (1/2)(\partial_t \Phi^\mu - \partial_z \Phi^\mu) = 0, \tag{3.93}$$

where the components of Φ^μ are given by (3.89). The four PDEs obtained after carrying out the derivatives are

$$b + l_+(l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta a + l_-(l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta b = 0, \tag{3.94a}$$

$$\partial_t c - \partial_z c = 0, \tag{3.94b}$$

$$\partial_t d - \partial_z d = 0, \tag{3.94c}$$

$$-b + l_+(l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta a - l_-(l_-/|l_-|)\mathrm{e}^{-2y/\lambda}\partial_\beta b = 0. \tag{3.94d}$$

Since (3.94b) and (3.94c) are the same PDEs as in the scalar field case, c and d must both be independent of β. We can easily decouple (3.94a) and (3.94d) by adding or subtracting them. Adding and discarding the non-zero terms gives

$$l_+\partial_\beta a = 0, \tag{3.95}$$

which requires $\partial_\beta a = 0$ everywhere, except possibly at $l_+ = 0$. Ignoring this technicality, we can say that a must be a function of α and x only. Subtracting the two PDEs and rearranging gives

$$\partial_\beta b(\alpha, \beta, x) = -b(\alpha, \beta, x)/\beta, \tag{3.96}$$

which has the solution

$$b(\alpha, \beta, x) = b(\alpha, x)/\beta. \tag{3.97}$$

Our $\widetilde{B}_\lambda$, X_1 invariant four vector field is therefore

$$\Phi^\mu = \begin{pmatrix} l_+a + (l_-/\beta)b \\ c \\ d \\ l_+a - (l_-/\beta)b \end{pmatrix}, \tag{3.98}$$

where a, b, c and d are functions of α and x.

3.5.3 $\widetilde{B}_\lambda$, X_1, X_3 invariant fields

The invariance condition for scalar fields is now

$$\mathcal{L}_{X_3}\phi(\alpha, x) = \xi_{X_3}{}^\sigma\partial_\sigma\phi(\alpha, x) = \partial_x\phi = 0, \tag{3.99}$$

implying that ϕ must be independent of x, a function of the single variable α. The invariance condition for four vector fields is

$$\mathcal{L}_{X_3}\Phi^\mu = \xi_{X_3}{}^\sigma\partial_\sigma\Phi^\mu - (\partial_\sigma\xi_{X_3}{}^\mu)\Phi^\sigma = \partial_x\Phi^\mu = 0, \tag{3.100}$$

where the components Φ^μ are given in (3.98). The four PDEs for each component are

$$l_+\partial_x a + (l_-/\beta)\partial_x b = 0, \tag{3.101a}$$

$$\partial_x c = 0, \tag{3.101b}$$

$$\partial_x d = 0, \tag{3.101c}$$

$$l_+\partial_x a - (l_-/\beta)\partial_x b = 0. \tag{3.101d}$$

The middle two equations obviously imply that c and d are functions of α only. Adding the first and last equations gives

$$l_+\partial_x a = 0. \tag{3.102}$$

Again ignoring the ambiguity at the $l_+ = 0$ point, we conclude that a is a function of α only. Subtracting the last PDE from the first gives us

$$(l_-/\beta)\partial_x b = (l_-/|l_-|)\mathrm{e}^{2y/\lambda}\partial_x b = 0, \tag{3.103}$$

implying that that b is a function of α only. The $\widetilde{B}_\lambda$, X_1, X_3 invariant form of the four vector field is

$$\Phi^\mu = \begin{pmatrix} l_+a + (l_-/\beta)b \\ c \\ d \\ l_+a - (l_-/\beta)b \end{pmatrix}, \tag{3.104}$$

where a, b, c and d are functions of α only.

3.5.4 $\widetilde{B}_\lambda$, X_1, X_3, X_4 invariant fields

Lastly, for full $\widetilde{P}_{13,10}$ invariance, the scalar field must satisfy

$$\mathcal{L}_{X_4}\phi(\alpha) = \xi_{X_4}{}^\sigma\partial_\sigma\phi(\alpha) = (1/2)(\partial_t\phi + \partial_z\phi) = 0. \tag{3.105}$$

After applying the chain rule and disregarding the non-zero terms, we get

$$\partial_\alpha\phi = 0, \tag{3.106}$$

so ϕ is a constant. $\widetilde{P}_{13,10}$ invariant vector fields must satisfy

$$\mathcal{L}_{X_4}\Phi^\mu = \xi_{X_4}{}^\sigma\partial_\sigma\Phi^\mu - (\partial_\sigma\xi_{X_4}{}^\mu)\Phi^\sigma = (1/2)(\partial_t\Phi^\mu + \partial_z\Phi^\mu) = 0, \tag{3.107}$$

where Φ^μ has components (3.104). The PDEs for each component are

$$a + l_+(l_+/|l_+|)\mathrm{e}^{2y/\lambda}\partial_\alpha a + (L/|L|)\mathrm{e}^{4y/\lambda}\partial_\alpha b = 0, \tag{3.108a}$$
$$\partial_t c + \partial_z c = 0, \tag{3.108b}$$
$$\partial_t d + \partial_z d = 0, \tag{3.108c}$$
$$a + l_+(l_+/|l_+|)\mathrm{e}^{2y/\lambda}\partial_\alpha a - (L/|L|)\mathrm{e}^{4y/\lambda}\partial_\alpha b = 0. \tag{3.108d}$$

The middle two PDEs are the same as the scalar field case, so c and d are both constants. Adding the first and last equations and rearranging gives

$$\partial_\alpha a(\alpha) = -a(\alpha)/\alpha, \tag{3.109}$$

which has the solution

$$a(\alpha) = a/\alpha, \tag{3.110}$$

where the a on the right-hand side is a constant. Finally, subtracting the last PDE from the first, we find that

$$\partial_\alpha b = 0, \tag{3.111}$$

so b is a constant. The $\widetilde{P}_{13,10}$ invariant form of the four vector field is

$$\Phi^\mu = \begin{pmatrix} (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}a + (l_-/|l_-|)\mathrm{e}^{2y/\lambda}b \\ c \\ d \\ (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}a - (l_-/|l_-|)\mathrm{e}^{2y/\lambda}b \end{pmatrix}. \tag{3.112}$$

Chapter 4

Maxwell-Dirac Symmetry Reductions

In this chapter, we apply the Poincaré symmetry subgroups from chapter 3 to the Maxwell-Dirac equations (2.44)-(2.48), and observe how the system reduces under these constraints. By substituting (2.46) into (2.47), and subsequently substituting (2.47) into (2.48), we find that the system depends only on the two four vector fields j^μ and k^μ, and the two scalar fields σ and ω. Strictly speaking, k^μ is a pseudovector, and ω is a pseudoscalar, which means their sign under a Lorentz transformation depends on the determinant of the transforming matrix. The sign is negative for improper transformations, but since we are only dealing with symmetry under Lorentz transformations connected to the identity, we shall treat k^μ as a regular four vector field and ω as a scalar. It can explicitly be shown in the spherical symmetry case, that if one interprets k^μ in terms of a rank-3 tensor $T_{\nu\sigma\rho}$ fully contracted with a rank-4 Levi-Civita symbol $\epsilon^{\mu\nu\sigma\rho}$, and then imposes the symmetric form of $T_{\nu\sigma\rho}$, k^μ has the correct four-vector form for that symmetry group. In general, since the axial vector can be written in the form

$$k^\mu = \epsilon^{\mu\nu\sigma\rho} T_{\nu\sigma\rho}, \tag{4.1}$$

taking the Lie derivative of both sides, and assuming that $T_{\nu\sigma\rho}$ is invariant such that $\mathcal{L}_{\boldsymbol{\xi}}(\boldsymbol{T}) = 0$, implies that $\mathcal{L}_{\boldsymbol{\xi}}(\boldsymbol{k}) = 0$ because the Levi-Civita symbol is invariant under the identity connected component of the Poincaré group. The Fierz identities (2.44) and (2.45) can be used to eliminate two dependent variables from the symmetry reduced form, which in our analysis we choose to be from the k^μ four vector field.

Our analysis will proceed as follows. For each symmetry subgroup, we will restate the forms that the fields must take, then use the Fierz identities to obtain expressions for two of the dependent functions in k^μ in terms of other dependent functions. These expressions are used to eliminate the two functions from the system entirely, except in the cylindrical case, where this would unnecessary complicate things. Next, we reduce B^μ (2.46) for $\mu = 0 - 3$ by substituting in the subgroup-invariant forms for the fields. Due to the length of the calculations in the spherical and cylindrical cases, we will just state the results, where we find that B^μ has the correct form for a subgroup-invariant four vector field, with the dependent functions in terms of j^μ, k^μ, σ and ω. In order to save space, we introduce the more abbreviated derivative notation $\partial_t\sigma \equiv \sigma_t$ and $\partial_r j_a \equiv j_{a,r}$.

Following this, we reduce $F^{\mu\nu}$ (2.47) by substituting the subgroup-invariant forms of the fields, as well as our previously obtained B^μ. In the spherically and cylindrically symmetric cases, the B^μ components are quite long, so we enlist the aid of Mathematica to carry out the derivatives and factorization, along with some further manual manipulation. The results of the manual manipulation can be checked for errors by comparing with the original Mathematica output. Once the form of $F^{\mu\nu}$ has been obtained, we can substitute it into Maxwell's equations (2.48), yielding up to four equations, purely in terms of j^μ, k^μ, σ and ω only; the Maxwell-Dirac equations. We shall see that the Maxwell-Dirac system varies wildly in complexity depending on what the chosen symmetry group is, from a simple algebraic system in the $P_{11,2}$ case, to the cylindrical case, which yields a coupled set of third-order, non-linear PDEs too long to easily write in closed form.

4.1 Spherical symmetry (subgroup $P_{3,4}$)

4.1.1 Fierz identities

From (3.29) in section 4.2, j^μ and k^μ have the form

$$j^\mu = \begin{pmatrix} j_a \\ xj_b \\ yj_b \\ zj_b \end{pmatrix}, \qquad k^\mu = \begin{pmatrix} k_a \\ xk_b \\ yk_b \\ zk_b \end{pmatrix}, \tag{4.2}$$

where j_a, j_b, k_a and k_b are all functions of t and $r = \sqrt{x^2+y^2+z^2}$, as are σ and ω. Lowering the index of the four vectors turns the column into a row, and changes the sign of the $\mu = 1, 2, 3$ components. Contracting j^μ and k^μ with themselves, and using the Fierz identity (2.44), we get

$$j_a^2 - r^2 j_b^2 = -k_a^2 + r^2 k_b^2 = \sigma^2 - \omega^2. \tag{4.3}$$

We can rearrange this to solve for k_a

$$k_a = \pm\sqrt{r^2(j_b^2 + k_b^2) - j_a^2}. \tag{4.4}$$

Contracting j^μ with k^μ and using the orthogonality condition (2.45)

$$j_a k_a - r^2 j_b k_b = 0, \tag{4.5}$$

then substituting (4.4) and performing some simple algebraic manipulation gives us the identity

$$k_b = \pm j_a / r. \tag{4.6}$$

Substituting this back into (4.4) gives the other identity

$$k_a = \pm r j_b. \tag{4.7}$$

We have determined that k^μ can be expressed entirely in terms of the dependent functions of j^μ

$$k^\mu = \pm \begin{pmatrix} rj_b \\ (x/r)j_a \\ (y/r)j_a \\ (z/r)j_a \end{pmatrix}. \tag{4.8}$$

4.1.2 Vector potential

Our next step is to substitute our symmetric fields into (2.46), and simplify for each μ. When performing the calculations, dealing with each term in the numerator of B^μ separately makes them much easier to handle. We will briefly outline the calculations for the first two components, then skip to the symmetric form of B^μ. When $\mu = 0$, the first component is

$$\begin{aligned}
&\epsilon^{0\nu\rho\sigma}(\sigma^2-\omega^2)\partial_\nu(j_\rho k_\sigma) \\
&\quad= (\sigma^2-\omega^2)[\partial_1(j_2k_3)-\partial_1(j_3k_2)-\partial_2(j_1k_3)+\partial_2(j_3k_1)+\partial_3(j_1k_2)-\partial_3(j_2k_1)] \\
&\quad= \pm(\sigma^2-\omega^2)\{\partial_x[(yz/r)j_aj_b]-\partial_x[(yz/r)j_aj_b]-\partial_y[(xz/r)j_aj_b] \\
&\qquad\quad+\partial_y[(xz/r)j_aj_b]+\partial_z[(xy/r)j_aj_b]-\partial_z[(xy/r)j_aj_b]\} \\
&\quad= 0.
\end{aligned} \tag{4.9}$$

The second term in B^0 expands in exactly the same way, except the partial derivative operator acts on $\sigma^2-\omega^2$, so we get

$$-(1/2)\epsilon^{0\nu\rho\sigma}j_\rho k_\sigma\partial_\nu(\sigma^2-\omega^2) = 0. \tag{4.10}$$

The third term in B^0 is

$$\begin{aligned}
&\delta^{0\nu\rho\sigma}[(\partial_\nu\sigma)\omega-\sigma(\partial_\nu\omega)]j_\rho k_\sigma = \mathrm{i}(j^0k^\nu - j^\nu k^0)[(\partial_\nu\sigma)\omega-\sigma(\partial_\nu\omega)] \\
&\quad= \pm\mathrm{i}[rj_aj_b(\partial_t\sigma)\omega+(x/r)j_a^2(\partial_x\sigma)\omega+(y/r)j_a^2(\partial_y\sigma)\omega+(z/r)j_a^2(\partial_z\sigma)\omega \\
&\qquad\quad- rj_aj_b\sigma(\partial_t\omega)-(x/r)j_a^2\sigma(\partial_x\omega)-(y/r)j_a^2\sigma(\partial_y\omega)-(z/r)j_a^2\sigma(\partial_z\omega) \\
&\qquad\quad- rj_aj_b(\partial_t\sigma)\omega-rxj_b^2(\partial_x\sigma)\omega-ryj_b^2(\partial_y\sigma)\omega-rzj_b^2(\partial_z\sigma)\omega \\
&\qquad\quad+ rj_aj_b\sigma(\partial_t\omega)+rxj_b^2\sigma(\partial_x\omega)+ryj_b^2\sigma(\partial_y\omega)+rzj_b^2\sigma(\partial_z\omega)] \\
&\quad= \pm\mathrm{i}[(j_a^2/r)-rj_b^2](x^2/r+y^2/r+z^2/r)[(\partial_r\sigma)\omega-\sigma(\partial_r\omega)] \\
&\quad= \pm\mathrm{i}(\sigma^2-\omega^2)[(\partial_r\sigma)\omega-\sigma(\partial_r\omega)].
\end{aligned} \tag{4.11}$$

Substituting these terms back into B^0 gives, after canceling terms and applying our abbreviated notation

$$B^0 = [\pm(\mathrm{i}/2)(\sigma_r\omega-\sigma\omega_r)-m\sigma j_a][q(\sigma^2-\omega^2)]^{-1}. \tag{4.12}$$

Now take the $\mu = 1$ case. The first term in B^1 is

$$\begin{aligned}
&\epsilon^{1\nu\rho\sigma}(\sigma^2-\omega^2)\partial_\nu(j_\rho k_\sigma) \\
&\quad= (\sigma^2-\omega^2)[-\partial_0(j_2k_3)+\partial_0(j_3k_2)+\partial_2(j_0k_3)-\partial_2(j_3k_0)-\partial_3(j_0k_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \partial_3(j_2 k_0)] \\
&= \pm(\sigma^2 - \omega^2)[(yz/r^3)j_a^2 - (2yz/r^2)j_a(\partial_r j_a) + (yz/r)j_b^2 + 2yzj_b(\partial_r j_b) \\
&\quad - (yz/r^3)j_a^2 + (2yz/r^2)j_a(\partial_r j_a) - (yz/r)j_b^2 - 2yzj_b(\partial_r j_b)] \\
&= 0.
\end{aligned} \tag{4.13}
$$

Since we had to apply the derivative operators in this case, we must be more careful about the second term, which is

$$
\begin{aligned}
&-(1/2)\epsilon^{1\nu\rho\sigma} j_\rho k_\sigma \partial_\nu(\sigma^2 - \omega^2) \\
&\quad = \pm(1/2)[(yz/r)j_a j_b \partial_t - (yz/r)j_a j_b \partial_t + (z/r)j_a^2 \partial_y - zrj_b^2 \partial_y - (y/r)j_a^2 \partial_z \\
&\qquad + yrj_b^2 \partial_z](\sigma^2 - \omega^2) \\
&\quad = \pm(1/2)[(2yz/r^2)j_a^2 - 2yzj_b^2 - (2yz/r^2)j_a^2 + 2yzj_b^2][\sigma(\partial_r \sigma) - \omega(\partial_r \omega)] \\
&\quad = 0.
\end{aligned} \tag{4.14}
$$

Taking advantage of the abbreviated derivative notation, the third term in B^1 is

$$
\begin{aligned}
\delta^{1\nu\rho\sigma}[(\partial_\nu \sigma)\omega - \sigma(\partial_\nu \omega)]j_\rho k_\sigma &= \mathrm{i}(j^1 k^\nu - j^\nu k^1)[(\partial_\nu \sigma)\omega - \sigma(\partial_\nu \omega)] \\
&= \pm\mathrm{i}[xrj_b^2(\sigma_t \omega - \sigma\omega_t) + (xy/r)j_a j_b(\sigma_y \omega - \sigma\omega_y) \\
&\quad + (xz/r)j_a j_b(\sigma_z \omega - \sigma\omega_z) - (x/r)j_a^2(\sigma_t \omega - \sigma\omega_t) \\
&\quad - (xy/r)j_a j_b(\sigma_y \omega - \sigma\omega_y) - (xz/r)j_a j_b(\sigma_z \omega - \sigma\omega_z)] \\
&= \mp\mathrm{i}(x/r)(\sigma^2 - \omega^2)(\sigma_t \omega - \sigma\omega_t).
\end{aligned} \tag{4.15}
$$

Substituting into B^1 and making appropriate cancellations gives

$$
B^1 = x[\mp(\mathrm{i}/2r)(\sigma_t \omega - \sigma\omega_t) - m\sigma j_b][q(\sigma^2 - \omega^2)]^{-1}. \tag{4.16}
$$

There is a similar result for $\mu = 2$ and $\mu = 3$, but instead of an x, there is a y and z respectively. We have found that B^μ assumes the form required for a spherically symmetric four vector field

$$
B^\mu = \begin{pmatrix} B_a \\ xB_b \\ yB_b \\ zB_b \end{pmatrix}, \tag{4.17}
$$

where B_a and B_b are functions of the invariants, given by

$$
B_a = [\pm(\mathrm{i}/2)(\sigma_r \omega - \sigma\omega_r) - m\sigma j_a][q(\sigma^2 - \omega^2)]^{-1}, \tag{4.18}
$$

$$
B_b = [\mp(\mathrm{i}/2r)(\sigma_t \omega - \sigma\omega_t) - m\sigma j_b][q(\sigma^2 - \omega^2)]^{-1}. \tag{4.19}
$$

4.1.3 Field strength tensor

Now we turn our attention to $F_{\mu\nu}$ (2.47), which is antisymmetric with six independent components. Since the calculations are quite lengthy, we enlist the computational aid of Mathematica. Considering the $\mu = 0$, $\nu = i = 1, 2, 3$ components of $F_{\mu\nu}$ first, we find that

$$
\epsilon^{\sigma\rho\kappa\tau} j_\kappa k_\tau[(\partial_0 j_\sigma)(\partial_i j_\rho) - (\partial_0 k_\sigma)(\partial_i k_\rho)] = 0, \tag{4.20}
$$

so the rational term vanishes. We are left with the four-curl term

$$F_{0i} = \partial_0 B_i - \partial_i B_0 = x_i[-\partial_t B_b - (1/r)\partial_r B_a], \tag{4.21}$$

which is of the form

$$F_{0i} = x_i F_a(t, r), \tag{4.22}$$

where the function of the invariants in terms of the bilinear fields is

$$\begin{aligned} F_a(t,r) = {} & (1/qr)(\sigma^2-\omega^2)^{-2}\{-2m[\sigma j_a(\sigma\sigma_r - \omega\omega_r) + r\sigma j_b(\sigma\sigma_t - \omega\omega_t)] \\ & \pm \mathrm{i}[\sigma\omega(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2) + (\sigma^2+\omega^2)(\sigma_t\omega_t - \sigma_r\omega_r)]\} \\ & + (1/qr)(\sigma^2-\omega^2)^{-1}[m(\sigma_r j_a + \sigma j_{a,r} + r\sigma_t j_b + r\sigma j_{b,t}) \\ & \pm (\mathrm{i}/2)(\sigma_{tt}\omega - \sigma\omega_{tt} - \sigma_{rr}\omega + \sigma\omega_{rr})]. \end{aligned} \tag{4.23}$$

Now consider the purely spatial components, F_{ij}. The four-curl term in this case is

$$\partial_i B_j - \partial_j B_i = -(x_j x_i/r)\partial_r B_b + (x_i x_j/r)\partial_r B_b = 0, \tag{4.24}$$

leaving us with the rational term only. Expanding the three independent F_{ij} using Mathematica, we find that

$$F_{12} = zF_b(t,r), \tag{4.25a}$$

$$F_{13} = -yF_b(t,r), \tag{4.25b}$$

$$F_{23} = xF_b(t,r), \tag{4.25c}$$

where

$$F_b(t,r) = \pm\frac{1}{2q}\left(\frac{j_a^4}{r^3} - \frac{2j_a^2 j_b^2}{r} + r j_b^4\right)(\sigma^2-\omega^2)^{-2}. \tag{4.26}$$

Factorizing, and using the inner product Fierz identity (4.3), we find that this simplifies to

$$F_b(t,r) = \pm\frac{1}{2qr^3}. \tag{4.27}$$

In summary, our spherically symmetric field strength tensor is of the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & xF_a & yF_a & zF_a \\ -xF_a & 0 & zF_b & -yF_b \\ -yF_a & -zF_b & 0 & xF_b \\ -zF_a & yF_b & -xF_b & 0 \end{pmatrix}. \tag{4.28}$$

The spatial components of the field strength tensor become

$$F_{12} = -M_z = \pm\frac{1}{2q}\frac{z}{r^3}, \tag{4.29a}$$

$$F_{13} = M_y = \mp\frac{1}{2q}\frac{y}{r^3}, \tag{4.29b}$$

$$F_{23} = -M_x = \pm\frac{1}{2q}\frac{x}{r^3}, \tag{4.29c}$$

implying a magnetic field of the form

$$\boldsymbol{M} = \mp\frac{1}{2q}\frac{\hat{\boldsymbol{r}}}{r^2}, \tag{4.30}$$

where we have unconventionally denoted the magnetic field by $\boldsymbol{M} = M_x\hat{\boldsymbol{x}} + M_y\hat{\boldsymbol{y}} + M_z\hat{\boldsymbol{z}}$ to avoid confusion with the gauge invariant vector potential. We can see that the $\boldsymbol{M}$-field is radially pointing and obeys an inverse square law, implying that there is a magnetic monopole at the origin. Taking the divergence of (4.30), we find that the only non-zero point is located at $r = 0$. Calling the magnetic charge q_m and the associated magnetic charge density ρ_m, we must have

$$\nabla \cdot \boldsymbol{M} = \rho_m = q_m\delta(r), \tag{4.31}$$

the volume integral of the right-hand side being equal to q_m. Applying the divergence theorem to the left-hand side of the volume integral of (4.31), we find that

$$\int \nabla \cdot \boldsymbol{M}\mathrm{d}V = \oint \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{a} = \mp\frac{2\pi}{q} = q_m, \tag{4.32}$$

which is in agreement with the Dirac quantization condition [14]

$$q_m q/4\pi = n/2, \tag{4.33}$$

for the special case where $n = \mp 1$.

This result follows simply from imposing spherical symmetry on our manifestly gauge invariant Dirac and Fierz formalism, and is a generalization of Radford's formalism [37], in that the monopole field appears in both the static *and* non-static cases. We shall see in the next section that the F_b dependent terms, and hence the magnetic monopole aspect, cancel out of the Maxwell equations, so monopoles have no direct effect on the physics of the coupled Maxwell-Dirac system. This is not to say that the presence of the monopole term is trivial; it is necessary for the spherical symmetry case to exist at all. Moreover, the monopole field (4.30) exists even in the case where there is no Dirac matter present ($j_a = 0$), which implies that it is an *external* electromagnetic field over all space, excluding the origin. Including the point $r = 0$ would imply a non-zero divergence of the magnetic field that is not consistent with our current scheme, which lacks dual magnetic charge terms. So despite our implicit exclusion of Maxwell fields not sourced by the Dirac matter, the imposition of spherical symmetry has caused their restricted inclusion indirectly, presumably to the extent necessary for the symmetry reduction to exist.

4.1.4 Maxwell equations

Combining the reduced field strength tensor (which was obtained from the inverted Dirac equation) with Maxwell's equations (2.48) results in the full Maxwell-Dirac system. It is easy to show that for the current symmetry group $SO(3)$, there are only two independent equations

$$3F_a + r\partial_r F_a = qj_a \tag{4.34}$$

$$\partial_t F_a = -qj_b, \tag{4.35}$$

for $\mu = 0$ and $\mu = i$ respectively. The F_b magnetic monopole terms appear in the $\mu = i$ equations, but cancel out, as previously discussed. Carrying out the derivatives with Mathematica and factorizing manually gives the two Maxwell-Dirac equations in terms of σ, ω, j_a, j_b and their t and r derivatives, up to third order. The first equation is

$$\begin{aligned}
q^2 j_a = {} & -(\sigma^2-\omega^2)^{-3} 4(\sigma\sigma_r - \omega\omega_r)\{-2m[\sigma j_a(\sigma\sigma_r - \omega\omega_r) + r\sigma j_b(\sigma\sigma_t - \omega\omega_t)] \\
& \pm \mathrm{i}[(\sigma^2+\omega^2)(\sigma_t\omega_t - \sigma_r\omega_r) + \sigma\omega(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2)]\} \\
& + (\sigma^2-\omega^2)^{-2}\{-2m[(3\sigma j_b + r\sigma_r j_b + r\sigma j_{b,r})(\sigma\sigma_t - \omega\omega_t) + (2\sigma_r j_a + 2\sigma j_{a,r} \\
& + 2\sigma j_a/r + r\sigma_t j_b + r\sigma j_{b,t})(\sigma\sigma_r - \omega\omega_r) + \sigma j_a(\sigma_r^2 + \sigma\sigma_{rr} - \omega_r^2 - \omega\omega_{rr}) \\
& + r\sigma j_b(\sigma\sigma_{tr} + \sigma_t\sigma_r - \omega\omega_{tr} - \omega_t\omega_r)] \pm \mathrm{i}[2(\sigma\sigma_r + \omega\omega_r + \sigma^2/r + \omega^2/r)(\sigma_t\omega_t \\
& - \sigma_r\omega_r) + (\sigma\omega_{tt} - \sigma_{tt}\omega - \sigma\omega_{rr} + \sigma_{rr}\omega)(\sigma\sigma_r - \omega\omega_r) + (\sigma\omega_r + \sigma_r\omega \\
& + 2\sigma\omega/r)(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2) + (\sigma^2+\omega^2)(\sigma_t\omega_{tr} + \sigma_{tr}\omega_t - \sigma_r\omega_{rr} - \sigma_{rr}\omega_r) \\
& + 2\sigma\omega(\sigma_r\sigma_{rr} - \sigma_t\sigma_{tr} + \omega_r\omega_{rr} - \omega_t\omega_{tr})]\} + (\sigma^2-\omega^2)^{-1}\{m(\sigma_{rr} j_a + 2\sigma_r j_{a,r} \\
& + \sigma j_{a,rr} + r\sigma_{tr} j_b + r\sigma_t j_{b,r} + r\sigma_r j_{b,t} + r\sigma j_{b,tr} + 3\sigma_t j_b + 3\sigma j_{b,t} + 2\sigma_r j_a/r \\
& + 2\sigma j_{a,r}/r) \pm \mathrm{i}[(1/2)(\sigma\omega_{rrr} + \sigma_r\omega_{rr} - \sigma_{rr}\omega_r - \sigma_{rrr}\omega - \sigma\omega_{ttr} - \sigma_r\omega_{tt} \\
& + \sigma_{tt}\omega_r + \sigma_{ttr}\omega) - (1/r)(\sigma\omega_{tt} - \sigma_{tt}\omega - \sigma\omega_{rr} + \sigma_{rr}\omega)]\},
\end{aligned} \tag{4.36}$$

and the second is

$$\begin{aligned}
q^2 r j_b = {} & (\sigma^2-\omega^2)^{-3} 4(\sigma\sigma_t - \omega\omega_t)\{-2m[\sigma j_a(\sigma\sigma_r - \omega\omega_r) + r\sigma j_b(\sigma\sigma_t - \omega\omega_t)] \\
& \pm \mathrm{i}[(\sigma^2+\omega^2)(\sigma_t\omega_t - \sigma_r\omega_r) + \sigma\omega(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2)]\} \\
& - (\sigma^2-\omega^2)^{-2}\{-2m[(\sigma j_{a,r} + \sigma_r j_a + 2r\sigma_t j_b + 2r\sigma j_{b,t})(\sigma\sigma_t - \omega\omega_t) \\
& + (\sigma j_{a,t} + \sigma_t j_a)(\sigma\sigma_r - \omega\omega_r) + \sigma j_a(\sigma\sigma_{tr} + \sigma_t\sigma_r - \omega\omega_{tr} - \omega_t\omega_r) + r\sigma j_b(\sigma_t^2 \\
& + \sigma\sigma_{tt} - \omega_t^2 - \omega\omega_{tt})] \pm \mathrm{i}[(\sigma_{rr}\omega - \sigma\omega_{rr} - \sigma_{tt}\omega + \sigma\omega_{tt})(\sigma\sigma_t - \omega\omega_t) \\
& + 2(\sigma\sigma_t + \omega\omega_t)(\sigma_t\omega_t - \sigma_r\omega_r) + (\sigma_t\omega + \sigma\omega_t)(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2) \\
& + (\sigma^2+\omega^2)(\sigma_{tt}\omega_t + \sigma_t\omega_{tt} - \sigma_{tr}\omega_r - \sigma_r\omega_{tr}) + 2\sigma\omega(\sigma_{tr}\sigma_r - \sigma_{tt}\sigma_t + \omega_{tr}\omega_r \\
& - \omega_{tt}\omega_t)]\} - (\sigma^2-\omega^2)^{-1}[m(\sigma_{tr} j_a + \sigma_t j_{a,r} + \sigma_r j_{a,t} + \sigma j_{a,tr} + r\sigma_{tt} j_b \\
& + 2r\sigma_t j_{b,t} + r\sigma j_{b,tt}) \pm (\mathrm{i}/2)(\sigma_{ttt}\omega + \sigma_{tt}\omega_t - \sigma_t\omega_{tt} - \sigma\omega_{ttt} - \sigma_{trr}\omega - \sigma_{rr}\omega_t \\
& + \sigma_t\omega_{rr} + \sigma\omega_{trr})].
\end{aligned} \tag{4.37}$$

Note that we still have the freedom to eliminate another field, by applying the Fierz identity

$$j_a^2 - r^2 j_b^2 = \sigma^2 - \omega^2. \tag{4.38}$$

Additionally, applying spherical symmetry to (2.49) and (2.50) we have the two respective physical constraint equations

$$j_{a,t} + 3j_b + r j_{b,r} = 0, \tag{4.39}$$

$$r j_{b,t} + (2/r) j_a + j_{a,r} = \mp 2\mathrm{i}m\omega. \tag{4.40}$$

4.2 Cylindrical symmetry (subgroup $P_{12,8}$)

4.2.1 Fierz identities

From (3.36), j^μ and k^μ take the form

$$j^\mu = \begin{pmatrix} j_a \\ xj_b - yj_c \\ yj_b + xj_c \\ j_d \end{pmatrix}, \quad k^\mu = \begin{pmatrix} k_a \\ xk_b - yk_c \\ yk_b + xk_c \\ k_d \end{pmatrix}, \tag{4.41}$$

where j_a, j_b, etc. are functions of t and $\rho = \sqrt{x^2+y^2}$. From (2.44), these forms imply

$$j_a^2 - \rho^2(j_b^2 + j_c^2) - j_d^2 = -k_a^2 + \rho^2(k_b^2 + k_c^2) + k_d^2 = \sigma^2 - \omega^2, \tag{4.42}$$

where σ and ω are both functions of t and ρ. Additionally, from (2.45) we have

$$j_a k_a - \rho^2(j_b k_b + j_c k_c) - j_d k_d = 0. \tag{4.43}$$

In the cylindrical case, there are four dependent functions in each four vector field (as opposed to two in the spherical case), so we can arbitrarily choose to eliminate two of them using the Fierz identities. Let us choose to solve for k_a and k_d as a single example. Rearranging (4.43) gives

$$k_a = [\rho^2(j_b k_b + j_c k_c) + j_d k_d] j_a^{-1}. \tag{4.44}$$

Substituting this into (4.42) and rearranging, we obtain the quadratic expression for k_d

$$\begin{aligned}(j_d^2 - j_a^2)k_d^2 + 2\rho^2 j_d(j_b k_b + j_c k_c)k_d + [j_a^4 - \rho^2 j_a^2(j_b^2 + j_c^2 + k_b^2 + k_c^2) \\ + \rho^4(j_b k_b + j_c k_c)^2 - j_a^2 j_d^2] = 0,\end{aligned} \tag{4.45}$$

which has the solution according to the quadratic formula

$$\begin{aligned}k_d = (-\rho^2 j_d(j_b k_b + j_c k_c) \pm \{\rho^4 j_d^2(j_b k_b + j_c k_c)^2 - (j_d^2 - j_a^2)[j_a^4 - \rho^2 j_a^2(j_b^2 + j_c^2 \\ + k_b^2 + k_c^2) + \rho^4(j_b k_b + j_c k_c)^2 - j_a^2 j_d^2]\}^{1/2})(j_d^2 - j_a^2)^{-1},\end{aligned} \tag{4.46}$$

after canceling out the factor of 2. Algebraic manipulation of the square root argument yields the simpler form

$$\begin{aligned}k_d = \{-\rho^2 j_d(j_b k_b + j_c k_c) \pm j_a[(j_a^2 - j_d^2)^2 - \rho^2(j_a^2 - j_d^2)(j_b^2 + j_c^2 + k_b^2 + k_c^2) \\ + \rho^4(j_b k_b + j_c k_c)^2]^{1/2}\}(j_d^2 - j_a^2)^{-1},\end{aligned} \tag{4.47}$$

which can be substituted into (4.44) to give

$$\begin{aligned}k_a = \{-\rho^2 j_a(j_b k_b + j_c k_c) \pm j_d[(j_a^2 - j_d^2)^2 - \rho^2(j_a^2 - j_d^2)(j_b^2 + j_c^2 + k_b^2 + k_c^2) \\ + \rho^4(j_b k_b + j_c k_c)^2]^{1/2}\}(j_d^2 - j_a^2)^{-1}.\end{aligned} \tag{4.48}$$

Unlike the spherical case, the Fierz identities do not provide a tidy replacement of the components of k^μ, so in our calculation of the reduced Maxwell-Dirac equations, we will retain all of the k^μ dependent functions with the implicit understanding that two of them can be eliminated.

4.2.2 Vector potential

As in the spherical case, we will look at each term in the numerator of in first rational term in (2.46) separately, then take the sum. Unlike in the spherical case, we do not show any of the calculation steps, as they are too lengthy, but not difficult. The four components of the vector potential are

$$\begin{aligned}
B^0 &= \{[\rho(j_{c,\rho}k_d + j_c k_{d,\rho} - j_{d,\rho}k_c - j_d k_{c,\rho}) + 2(j_c k_d - j_d k_c) - 2m\sigma j_a](\sigma^2 - \omega^2) \\
&\quad + \rho(j_d k_c - j_c k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}\rho(j_a k_b - j_b k_a)(\sigma_\rho\omega - \sigma\omega_\rho)\} \\
&\quad \cdot [2q(\sigma^2 - \omega^2)^2]^{-1}, &\text{(4.49a)} \\
B^1 &= x[(j_{d,t}k_c + j_d k_{c,t} - j_{c,t}k_d - j_c k_{d,t} - 2m\sigma j_b)(\sigma^2 - \omega^2) \\
&\quad + (j_c k_d - j_d k_c)(\sigma\sigma_t - \omega\omega_t) + \mathrm{i}(j_b k_a - j_a k_b)(\sigma_t\omega - \sigma\omega_t)][2q(\sigma^2 - \omega^2)^2]^{-1} \\
&\quad - y\{[j_{b,t}k_d + j_b k_{d,t} - j_{d,t}k_b - j_d k_{b,t} + (1/\rho)(j_{a,\rho}k_d + j_a k_{d,\rho} - j_{d,\rho}k_a \\
&\quad - j_d k_{a,\rho}) - 2m\sigma j_c](\sigma^2 - \omega^2) + (j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) \\
&\quad + (1/\rho)(j_d k_a - j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}(j_c k_a - j_a k_c)(\sigma_t\omega - \sigma\omega_t) \\
&\quad + \mathrm{i}\rho(j_c k_b - j_b k_c)(\sigma_\rho\omega - \sigma\omega_\rho)\}[2q(\sigma^2 - \omega^2)^2]^{-1}, &\text{(4.49b)} \\
B^2 &= y[(j_{d,t}k_c + j_d k_{c,t} - j_{c,t}k_d - j_c k_{d,t} - 2m\sigma j_b)(\sigma^2 - \omega^2) \\
&\quad + (j_c k_d - j_d k_c)(\sigma\sigma_t - \omega\omega_t) + \mathrm{i}(j_b k_a - j_a k_b)(\sigma_t\omega - \sigma\omega_t)][2q(\sigma^2 - \omega^2)^2]^{-1} \\
&\quad + x\{[j_{b,t}k_d + j_b k_{d,t} - j_{d,t}k_b - j_d k_{b,t} + (1/\rho)(j_{a,\rho}k_d + j_a k_{d,\rho} - j_{d,\rho}k_a \\
&\quad - j_d k_{a,\rho}) - 2m\sigma j_c](\sigma^2 - \omega^2) + (j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) \\
&\quad + (1/\rho)(j_d k_a - j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}(j_c k_a - j_a k_c)(\sigma_t\omega - \sigma\omega_t) \\
&\quad + \mathrm{i}\rho(j_c k_b - j_b k_c)(\sigma_\rho\omega - \sigma\omega_\rho)\}[2q(\sigma^2 - \omega^2)^2]^{-1}, &\text{(4.49c)} \\
B^3 &= \{[\rho^2(j_{c,t}k_b + j_c k_{b,t} - j_{b,t}k_c - j_b k_{c,t}) + \rho(j_{c,\rho}k_a + j_c k_{a,\rho} - j_{a,\rho}k_c - j_a k_{c,\rho}) \\
&\quad + 2(j_c k_a - j_a k_c) - 2m\sigma j_d](\sigma^2 - \omega^2) + \rho^2(j_b k_c - j_c k_b)(\sigma\sigma_t - \omega\omega_t) \\
&\quad + \rho(j_a k_c - j_c k_a)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}(j_d k_a - j_a k_d)(\sigma_t\omega - \sigma\omega_t) \\
&\quad + \mathrm{i}\rho(j_d k_b - j_b k_d)(\sigma_\rho\omega - \sigma\omega_\rho)\}[2q(\sigma^2 - \omega^2)^2]^{-1}. &\text{(4.49d)}
\end{aligned}$$

From inspection, we can see that the gauge invariant vector potential assumes the correct form for a cylindrically symmetric four vector field

$$B^\mu = \begin{pmatrix} B_a \\ xB_b - yB_c \\ yB_b + xB_c \\ B_d \end{pmatrix}, \tag{4.50}$$

where B_a, etc. are functions of the invariants t and ρ, and are defined as

$$\begin{aligned}
B_a &= \{[\rho(j_{c,\rho}k_d + j_c k_{d,\rho} - j_{d,\rho}k_c - j_d k_{c,\rho}) + 2(j_c k_d - j_d k_c) - 2m\sigma j_a](\sigma^2 - \omega^2) \\
&\quad + \rho(j_d k_c - j_c k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}\rho(j_a k_b - j_b k_a)(\sigma_\rho\omega - \sigma\omega_\rho)\} \\
&\quad \cdot [2q(\sigma^2 - \omega^2)^2]^{-1}, &\text{(4.51a)} \\
B_b &= [(j_{d,t}k_c + j_d k_{c,t} - j_{c,t}k_d - j_c k_{d,t} - 2m\sigma j_b)(\sigma^2 - \omega^2)
\end{aligned}$$

$$+ (j_c k_d - j_d k_c)(\sigma\sigma_t - \omega\omega_t) + \mathrm{i}(j_b k_a - j_a k_b)(\sigma_t\omega - \sigma\omega_t)][2q(\sigma^2 - \omega^2)^2]^{-1}, \tag{4.51b}$$

$$\begin{aligned} B_c = \{&[j_{b,t}k_d + j_b k_{d,t} - j_{d,t}k_b - j_d k_{b,t} + (1/\rho)(j_{a,\rho}k_d + j_a k_{d,\rho} - j_{d,\rho}k_a - j_d k_{a,\rho}) \\ &- 2m\sigma j_c](\sigma^2 - \omega^2) + (j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) + (1/\rho)(j_d k_a \\ &- j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}(j_c k_a - j_a k_c)(\sigma_t\omega - \sigma\omega_t) + \mathrm{i}\rho(j_c k_b - j_b k_c)(\sigma_\rho\omega \\ &- \sigma\omega_\rho)\}[2q(\sigma^2 - \omega^2)^2]^{-1}, \end{aligned} \tag{4.51c}$$

$$\begin{aligned} B_d = \{&[\rho^2(j_{c,t}k_b + j_c k_{b,t} - j_{b,t}k_c - j_b k_{c,t}) + \rho(j_{c,\rho}k_a + j_c k_{a,\rho} - j_{a,\rho}k_c - j_a k_{c,\rho}) \\ &+ 2(j_c k_a - j_a k_c) - 2m\sigma j_d](\sigma^2 - \omega^2) + \rho^2(j_b k_c - j_c k_b)(\sigma\sigma_t - \omega\omega_t) \\ &+ \rho(j_a k_c - j_c k_a)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}(j_d k_a - j_a k_d)(\sigma_t\omega - \sigma\omega_t) \\ &+ \mathrm{i}\rho(j_d k_b - j_b k_d)(\sigma_\rho\omega - \sigma\omega_\rho)\}[2q(\sigma^2 - \omega^2)^2]^{-1}. \end{aligned} \tag{4.51d}$$

4.2.3 Field strength tensor

We approach the calculation of $F_{\mu\nu}$ as we did the vector potential, in that we calculate the four-curl of B^μ and the rational term in (2.47) separately, then sum them together. Obviously, these calculations would be very time consuming to do by hand, so we use Mathematica to carry out the expansions, and manually factorizing. Due to the size of these expressions, their explicit form in terms of j^μ, k^μ, σ and ω are relegated to appendix D. We can calculate the form of the four curl of B^μ by substituting (4.50) into (2.47), which can in turn be expressed in terms of j^μ, etc. by applying (4.51a)-(4.51d). The $\mu = 0$, $\nu = 1$ four curl term is

$$\partial_0 B_1 - \partial_1 B_0 = \partial_t(-xB_b + yB_c) - \partial_x B_a = -x[\partial_t B_b + (1/\rho)\partial_\rho B_a] + y\partial_t B_c, \tag{4.52}$$

and if we take account of the rational term, we find that

$$F_{01} = -xF_a + yF_b, \tag{4.53}$$

where F_a, F_b, etc., are functions of t and ρ. The $\mu = 0$, $\nu = 2$ four curl term is

$$\partial_0 B_2 - \partial_2 B_0 = \partial_t(-yB_b - xB_c) - \partial_y B_a = -y[\partial_t B_b + (1/\rho)\partial_\rho B_a] - x\partial_t B_c, \tag{4.54}$$

which when including the rational term, gives

$$F_{02} = -yF_a - xF_b. \tag{4.55}$$

Now, considering the form of the rational term in F_{03}, we can see that $\partial_3 j_\sigma$ and $\partial_3 k_\sigma$ cause the entire term to vanish, due to z-translation invariance. The $\partial_3 B_0$ part of the four curl vanishes for the same reason, so we are left with

$$F_{03} = \partial_0 B_3 = \partial_t B_d = -F_c. \tag{4.56}$$

The $\mu = 1$, $\nu = 2$ four curl term is

$$\partial_1 B_2 - \partial_2 B_1 = \partial_x(-yB_b - xB_c) - \partial_y(-xB_b + yB_c) = -2B_c - \rho\partial_\rho B_c, \tag{4.57}$$

which when including the rational term, is of the form

$$F_{12} = F_d. \tag{4.58}$$

The rational terms in F_{13} and F_{23} are both zero for the same reason as in F_{03}, so we are just left with the four curl term in both cases. In the $\mu = 1$, $\nu = 3$ case we have

$$F_{13} = \partial_1 B_3 = (x/\rho)\partial_\rho B_d = xF_e, \tag{4.59}$$

and when $\mu = 2$, $\nu = 3$, we have

$$F_{23} = \partial_2 B_3 = (y/\rho)\partial_\rho B_d = yF_e. \tag{4.60}$$

So our field strength tensor form for cylindrical symmetry is

$$F_{\mu\nu} = \begin{pmatrix} 0 & -xF_a + yF_b & -yF_a - xF_b & -F_c \\ xF_a - yF_b & 0 & F_d & xF_e \\ yF_a + xF_b & -F_d & 0 & yF_e \\ F_c & -xF_e & -yF_e & 0 \end{pmatrix}, \tag{4.61}$$

where the forms of F_a, etc. in terms of j^μ, k^μ, σ and ω are given in (D.1)-(D.5).

4.2.4 Maxwell equations

Substituting (4.41) and (4.61) into (2.48), it is easy to obtain the four equations

$$qj_a = 2F_a + \rho\partial_\rho F_a, \tag{4.62a}$$
$$q(xj_b - yj_c) = x(-\partial_t F_a) - y[-\partial_t F_b + (1/\rho)\partial_\rho F_d], \tag{4.62b}$$
$$q(yj_b + xj_c) = y(-\partial_t F_a) + x[-\partial_t F_b + (1/\rho)\partial_\rho F_d], \tag{4.62c}$$
$$qj_d = 2F_e - \partial_t F_c + \rho\partial_\rho F_e. \tag{4.62d}$$

If we multiply (4.62b) by x and add (4.62c) multiplied by y, we obtain

$$qj_b = -\partial_t F_a. \tag{4.63}$$

Likewise, if we take the combination x times (4.62c) and subtract y times (4.62b), we get

$$qj_c = -\partial_t F_b + (1/\rho)\partial_\rho F_d. \tag{4.64}$$

The Maxwell equations for cylindrical symmetry therefore reduces to the set (4.62a), (4.62d), (4.63) and (4.64), equations dependent only on functions of t and ρ. These expressions are far too long to write explicitly, even in the appendix, but the full Maxwell-Dirac equations can be obtained simply by substituting in the expressions (D.1)-(D.5) from appendix D. In addition to the four Maxwell-Dirac equations, we have the three equations provided by the Fierz identities (4.42) and (4.43), as well as the continuity equations

$$2j_b + \partial_t j_a + \rho\partial_\rho j_b = 0, \tag{4.65a}$$
$$2k_b + \partial_t k_a + \rho\partial_\rho k_b = -2\mathrm{i}m\omega, \tag{4.65b}$$

obtained by applying the cylindrical four vector forms to (2.49) and (2.50).

4.3 $P_{11,2}$ symmetry ("screw" subgroup)

4.3.1 Fierz identities

From (3.74), j^μ and k^μ have the form

$$j^\mu = \begin{pmatrix} l_+ j_a \\ 0 \\ 0 \\ l_+ j_a \end{pmatrix}, \quad k^\mu = \begin{pmatrix} l_+ k_a \\ 0 \\ 0 \\ l_+ k_a \end{pmatrix}, \tag{4.66}$$

where j_a and k_a are constants. Scalar fields σ and ω are also constants. Applying these forms to the inner product Fierz identity (2.44) results in the expression

$$\sigma^2 - \omega^2 = 0, \tag{4.67}$$

because our $P_{11,2}$ invariant four vectors are null. The orthogonality condition (2.45) results in $0 = 0$, providing no further information.

4.3.2 Vector potential

Ignoring the fact that $\sigma^2 - \omega^2 = 0$ for the moment, we shall carry out the Maxwell-Dirac reduction to check what happens. Consider the gauge invariant vector potential (2.46). Since σ and ω are constants, all of the derivative terms involving them vanish. The $\epsilon^{\mu\nu\rho\sigma}\partial_\nu(j_\rho k_\sigma)$ term also vanishes due to antisymmetry because the only non-zero derivatives are for $\nu = 0$ when $\rho = 0$, $\sigma = 3$ and vice-versa. We are left with

$$B^\mu = -\frac{1}{q}\frac{m\sigma j^\mu}{\sigma^2 - \omega^2}, \tag{4.68}$$

which in explicit component form is

$$B^\mu = \begin{pmatrix} -l_+ m\sigma j_a / q(\sigma^2 - \omega^2) \\ 0 \\ 0 \\ -l_+ m\sigma j_a / q(\sigma^2 - \omega^2) \end{pmatrix}. \tag{4.69}$$

4.3.3 Field strength tensor

The rational term in (2.47) vanishes due to the antisymmetry of $\epsilon^{\sigma\rho\kappa\tau}$ and the fact that there are only two non-zero B^μ components, so we are just left with the four-curl term. The only non-zero component is $F_{03} = -F_{30}$, which is

$$F_{03} = \partial_t B_3 - \partial_z B_0 = 2m\sigma j_a / q(\sigma^2 - \omega^2), \tag{4.70}$$

a constant term. Since the field strength tensor is constant the left-hand side of the Maxwell equations (2.48) vanishes, leaving us with the result

$$j_a = 0. \tag{4.71}$$

4.3.4 Continuity equations

Now consider the two continuity equations (2.49) and (2.50). Applying our $P_{11,2}$ invariant forms, we find that

$$j_a = 0, \tag{4.72a}$$
$$k_a = -\mathrm{i}m\omega, \tag{4.72b}$$

the first equation confirming our Maxwell-Dirac result. Let us consider the result $\sigma^2 - \omega^2 = 0$ more closely. This can be rearranged to give $\sigma = \pm\omega$, but since ω is pure imaginary and σ is real, the only case in which they can be equal is when they are both zero. This in turn means that $k_a = 0$ also. We have thus obtained a closed form solution to the Maxwell-Dirac equations under $P_{11,2}$ symmetry, which unfortunately is constrained to be the trivial solution

$$\begin{aligned} &\sigma = \omega = 0, \\ &j^\mu = k^\mu = \underline{0}. \end{aligned} \tag{4.73}$$

Note that this solution was obtained using only the Fierz identities and continuity equations; it is unnecessary to deal with the full Maxwell-Dirac equations in this case.

4.4 $\widetilde{P}_{13,10}$ symmetry ("trans-boost" subgroup)

4.4.1 Fierz identities

From (3.112), the $\widetilde{P}_{13,10}$ invariant form of j^μ is

$$j^\mu = \begin{pmatrix} (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}j_a + (l_-/|l_-|)\mathrm{e}^{2y/\lambda}j_b \\ j_c \\ j_d \\ (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}j_a - (l_-/|l_-|)\mathrm{e}^{2y/\lambda}j_b \end{pmatrix}, \tag{4.74}$$

where j_a, j_b, etc. are constants. The axial four vector k^μ has the same form, but with k_a replacing j_a, and so on. Remember that $\lambda > 0$ is a continuous parameter associated with the $\widetilde{B}_\lambda$ generator, with each value representing a different symmetry. Applying the symmetric forms of j^μ and k^μ to the Fierz identities (2.44) and (2.45) gives

$$4(L/|L|)j_a j_b - j_c^2 - j_d^2 = -4(L/|L|)k_a k_b + k_c^2 + k_d^2 = \sigma^2 - \omega^2, \tag{4.75}$$
$$2(L/|L|)(j_a k_b + j_b k_a) - j_c k_c - j_d k_d = 0, \tag{4.76}$$

where $L \equiv l_+ l_-$ Unlike previous examples, we will not apply these immediately to replace elements of k^μ, but wait until the Maxwell-Dirac system has been obtained.

4.4.2 Vector potential

Since σ and ω are constants, the derivatives of these objects in (2.46) vanish, leaving us with

$$B^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}\partial_\nu(j_\rho k_\sigma) - 2m\sigma j^\mu}{2q(\sigma^2-\omega^2)}. \tag{4.77}$$

The four-vectors only vary in the $y-$direction, so the only non-zero derivatives are for $\nu = 2$ when $\rho = 0$, $\sigma = 3$ or $\rho = 3$, $\sigma = 0$. Setting $\mu = 0$, the first term in the numerator is

$$\begin{aligned}\epsilon^{02\rho\sigma}\partial_2(j_\rho k_\sigma) &= \epsilon^{0213}\partial_2(j_1k_3) + \epsilon^{0231}\partial_2(j_3k_1)\\ &= (l_+/|l_+|)(2/\lambda)\mathrm{e}^{-2y/\lambda}(j_ck_a - j_ak_c) + (l_-/|l_-|)(2/\lambda)\mathrm{e}^{2y/\lambda}(j_ck_b - j_bk_c).\end{aligned} \tag{4.78}$$

Substituting into (4.77) for $\mu = 0$, we get

$$B^0 = (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}B_a + (l_-/|l_-|)\mathrm{e}^{2y/\lambda}B_b, \tag{4.79}$$

where B_a and B_b are the constants

$$B_a = \frac{(1/\lambda)(j_ck_a - j_ak_c) - m\sigma j_a}{q(\sigma^2-\omega^2)}, \tag{4.80}$$

$$B_b = \frac{(1/\lambda)(j_ck_b - j_bk_c) - m\sigma j_b}{q(\sigma^2-\omega^2)}. \tag{4.81}$$

Setting $\mu = 1$ and $\mu = 2$, we find that the first numerator term in (4.77) vanishes in both cases, so these components are the constants

$$B^1 = B_c = -\frac{m\sigma j_c}{q(\sigma^2-\omega^2)}, \tag{4.82}$$

$$B^2 = B_d = -\frac{m\sigma j_d}{q(\sigma^2-\omega^2)}. \tag{4.83}$$

Lastly, we have $\mu = 3$. The first numerator term is

$$\begin{aligned}\epsilon^{32\rho\sigma}\partial_2(j_\rho k_\sigma) &= \epsilon^{3201}\partial_2(j_0k_1) + \epsilon^{3210}\partial_2(j_1k_0)\\ &= (l_+/|l_+|)(2/\lambda)\mathrm{e}^{-2y/\lambda}(j_ck_a - j_ak_c) - (l_-/|l_-|)(2/\lambda)\mathrm{e}^{2y/\lambda}(j_ck_b - j_bk_c),\end{aligned} \tag{4.84}$$

which when substituting into B^3 gives the final component

$$B^3 = (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}B_a - (l_-/|l_-|)\mathrm{e}^{2y/\lambda}B_b. \tag{4.85}$$

We have determined that B^μ has the correct form for a $\widetilde{P}_{13,10}$ invariant four vector field

$$B^\mu = \begin{pmatrix} (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}B_a + (l_-/|l_-|)\mathrm{e}^{2y/\lambda}B_b \\ B_c \\ B_d \\ (l_+/|l_+|)\mathrm{e}^{-2y/\lambda}B_a - (l_-/|l_-|)\mathrm{e}^{2y/\lambda}B_b \end{pmatrix}, \tag{4.86}$$

where B_a, etc. are constants, as required.

4.4.3 Field strength tensor

Consider the rational term in (2.47). Since the only non-constants in j_σ and k_σ are $e^{\pm 2y/\lambda}$, the only non-vanishing derivative is $\partial_2 \equiv \partial_y$. In general, the rational term must vanish, because $\mu \neq \nu$, so they cannot both be 2, and either derivative vanishing leads to the entire rational term vanishing. So the field strength tensor reduces to the four curl of B^μ for this symmetry group

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{4.87}$$

For similar reasons as above, the only non-vanishing elements of $F_{\mu\nu}$ are those where one of the indices is 2, and the other is either 0 or 3. Setting $\mu = 0$, $\nu = 2$, we get

$$F_{02} = \partial_0 B_2 - \partial_2 B_0 = (l_+/|l_+|)e^{-2y/\lambda} F_a - (l_-/|l_-|)e^{2y/\lambda} F_b, \tag{4.88}$$

where F_a and F_b are the constants

$$F_a = \frac{(2/\lambda)(j_c k_a - j_a k_c) - 2m\sigma j_a}{\lambda q(\sigma^2 - \omega^2)}, \tag{4.89}$$

$$F_b = \frac{(2/\lambda)(j_c k_b - j_b k_c) - 2m\sigma j_b}{\lambda q(\sigma^2 - \omega^2)}. \tag{4.90}$$

Setting $\mu = 2$, $\nu = 3$, we get

$$F_{23} = \partial_2 B_3 - \partial_3 B_2 = (l_+/|l_+|)e^{-2y/\lambda} F_a + (l_-/|l_-|)e^{2y/\lambda} F_b, \tag{4.91}$$

giving us the only other independent non-zero component of $F_{\mu\nu}$.

4.4.4 Maxwell equations

Setting $\mu = 0$ in (2.48), we get

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = q\left(\frac{l_+}{|l_+|}e^{-2y/\lambda} j_a + \frac{l_-}{|l_-|}e^{2y/\lambda} j_b\right). \tag{4.92}$$

The only non-zero term on the left hand side is $\partial_2 F^{20} = \partial_y F_{02}$, so carrying out the derivative and rearranging gives us

$$\begin{aligned}
&\frac{l_+}{|l_+|}e^{-2y/\lambda}\left\{\frac{4}{\lambda^2 q(\sigma^2-\omega^2)}\left[\frac{1}{\lambda}(j_c k_a - j_a k_c) - m\sigma j_a\right] + q j_a\right\} \\
&+ \frac{l_-}{|l_-|}e^{2y/\lambda}\left\{\frac{4}{\lambda^2 q(\sigma^2-\omega^2)}\left[\frac{1}{\lambda}(j_c k_b - j_b k_c) - m\sigma j_b\right] + q j_b\right\} = 0.
\end{aligned} \tag{4.93}$$

Setting $\mu = 1$ gives

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = q j_c. \tag{4.94}$$

Since the entire left hand side vanishes, we are left with

$$j_c = 0. \tag{4.95}$$

Setting $\mu = 2$, we find for the same reason that

$$j_d = 0. \tag{4.96}$$

Lastly, setting $\mu = 3$ gives

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = q\left(\frac{l_+}{|l_+|}\mathrm{e}^{-2y/\lambda} j_a - \frac{l_-}{|l_-|}\mathrm{e}^{2y/\lambda} j_b\right). \tag{4.97}$$

Recognizing that the only non-zero term on the left is $\partial_2 F^{23} = \partial_y F_{23}$, we end up with

$$\begin{aligned}&\frac{l_+}{|l_+|}\mathrm{e}^{-2y/\lambda}\left\{\frac{4}{\lambda^2 q(\sigma^2-\omega^2)}\left[\frac{1}{\lambda}(j_c k_a - j_a k_c) - m\sigma j_a\right] + q j_a\right\}\\ &-\frac{l_-}{|l_-|}\mathrm{e}^{2y/\lambda}\left\{\frac{4}{\lambda^2 q(\sigma^2-\omega^2)}\left[\frac{1}{\lambda}(j_c k_b - j_b k_c) - m\sigma j_b\right] + q j_b\right\} = 0.\end{aligned} \tag{4.98}$$

Adding (4.93) and (4.98), discarding the non-zero coefficient and applying (4.95), we get

$$\frac{4}{\lambda^2 q(\sigma^2-\omega^2)}\left(-\frac{j_a k_c}{\lambda} - m\sigma j_a\right) + q j_a = 0. \tag{4.99}$$

Canceling the common factor j_a and rearranging to solve for k_c gives

$$k_c = \frac{\lambda^3 q^2(\sigma^2-\omega^2)}{4} - \lambda m\sigma. \tag{4.100}$$

Subtracting (4.98) from (4.93), applying (4.95) then discarding the non-zero coefficient and the common factor j_b gives exactly (4.100). Therefore, the three equations (4.95), (4.96) and (4.100) constitute the Maxwell-Dirac equations for the $\widetilde{P}_{13,10}$ subalgebra.

4.4.5 Fierz-Maxwell-Dirac reduction

From this point on, for simplicity we shall set all of the discontinuous factors $L/|L| = 1$, choosing the positive sign. We can further simplify the k_c expression if we take into account the partial conservation of the axial four vector (2.50). Since the only non-zero derivatives of k^μ are $\partial_2 k^0$ and $\partial_2 k^3$, the left hand side vanishes, so the consistency condition

$$\omega = 0 \tag{4.101}$$

must hold. So (4.100) simplifies to an expression quadratic in σ

$$k_c = \frac{\lambda^3 q^2 \sigma^2}{4} - \lambda m\sigma. \tag{4.102}$$

Note that the continuity equation (2.49) gives the redundant expression $0 = 0$, as the left side vanishes for the same reasons as $\partial_\mu k^\mu$. Now consider the outer parts of the Fierz identity (4.75). Setting $j_c = j_d = \omega = 0$, this becomes

$$\sigma^2 = 4 j_a j_b, \tag{4.103}$$

which when substituted into (4.102) gives

$$k_c = \lambda^3 q^2 j_a j_b \mp 2\lambda m \sqrt{j_a j_b}. \tag{4.104}$$

Additionally, looking at the left hand parts of (4.75), we can see that setting $j_c = j_d = 0$ gives us

$$-4j_a j_b - 4k_a k_b + k_c^2 + k_d^2 = 0. \tag{4.105}$$

Setting $j_c = j_d = 0$ in the orthogonality Fierz identity (4.76) gives the additional relationship

$$j_a k_b = -j_b k_a. \tag{4.106}$$

Substituting (4.104) into (4.105) gives an algebraic equation of the form

$$f(j_a, j_b, k_a, k_b, k_d; \lambda) = 0, \tag{4.107}$$

where the function on the left is

$$f(j_a, j_b, k_a, k_b, k_d; \lambda) = \lambda^6 q^4 (j_a j_b)^2 \mp 4\lambda^4 q^2 m (j_a j_b)^{3/2} + 4(\lambda^2 m^2 - 1) j_a j_b - 4k_a k_b + k_d^2. \tag{4.108}$$

Values of j_a, j_b, k_a, k_b and k_d which solve (4.107) for a given λ constitute solutions to the Fierz-Maxwell-Dirac equations, symmetric under the $\widetilde{P}_{13,10}$ Poincaré subalgebra. Note that we can still eliminate one of the constants by imposing (4.106).

Chapter 5

Maxwell-Dirac stress-energy tensor

Now before moving on to deriving solutions to the reduced Maxwell-Dirac equations, we take a side step into more theoretical development, namely the bilinearization of the stress-energy tensor appropriate to the system. We perform this bilinearization in two independent ways. This provides us with a theoretical tool that can be used to calculate interesting physical quantities, such as mass-energy and momentum flux, for solutions obtained to the Maxwell-Dirac equations.

Firstly, we consider the Dirac spinor dependent part of the Belinfante tensor, a generalization of the well-known canonical form which is manifestly symmetric. What follows is a very similar calculation to that presented in section 2.3, whereby we derive a Fierz identity to replace the spinor terms in the Belinfante tensor with bilinear tensors. Adding the interaction and electromagnetic contributions to the resulting expression yields the bilinear form of the Maxwell-Dirac Belinfante tensor.

Using the known form of the Lagrangian density of an electromagnetically interacting Dirac particle, we can use a contracted form of the Belinfante Fierz identity to rewrite it in terms of bilinears. The variational approach of calculating the stress-energy tensor, known from general relativity, is then used to obtain a Maxwell-Dirac stress-energy tensor equivalent to the Belinfante form, thereby strengthening our result. Somewhat surprisingly, this agreement requires no consideration of the constraints involved in the spinor to bilinear mapping. Similar calculations have been performed by Rudolph and Kijowski [28], [29] in an alternative gauge invariant formulation of electrodynamics, where such functional constraints are included.

We conclude our discussion of the bilinear stress-energy tensor with an example symmetry reduction under the $SO(3)$ group. This is done in anticipation of the content of chapter 6, where we apply the stress-energy tensor to parametrize the mass-energy of obtained symmetric solutions.

5.1 Maxwell-Dirac stress-energy tensor via Belinfante

5.1.1 Belinfante tensor for a free Dirac particle

The Belinfante stress-energy tensor is the fully symmetric analogue of the well-known asymmetric "canonical" form, which for free Dirac particles is[1]

$$\widetilde{T}^{\mu\nu} = -\frac{\mathrm{i}}{2}\left[\overline{\psi}\gamma^{\mu}(\partial^{\nu}\psi) - (\partial^{\nu}\overline{\psi})\gamma^{\mu}\psi\right], \tag{5.1}$$

that satisfies the conservation condition

$$\partial_{\mu}\widetilde{T}^{\mu\nu} = 0. \tag{5.2}$$

In fact, $\widetilde{T}^{\mu\nu}$ is the Noether symmetry current corresponding to imposing the invariance of the free Dirac Lagrangian

$$\mathscr{L} = \frac{\mathrm{i}}{2}\left[\overline{\psi}\gamma^{\mu}(\partial_{\mu}\psi) - (\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi\right] - m\overline{\psi}\psi \tag{5.3}$$

under the translation group. The asymmetry arises from the fact that only translations are considered in the derivation of (5.1), which neglects rotational contributions to the stress-energy [15]. Since (5.3) is invariant under Lorentz transformations, we can use the formula for the Noether symmetry current divergence

$$\partial_{\mu}\left[\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\psi)}\delta\psi + \delta\overline{\psi}\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\overline{\psi})}\right] = 0, \tag{5.4}$$

to obtain the Noether current directly. A vanishing, manifestly antisymmetric expression is obtained

$$\frac{1}{4}\partial_{\sigma}\left(\overline{\psi}\{\gamma^{\sigma},\sigma^{\mu\nu}\}\psi\right) + \widetilde{T}^{\mu\nu} - \widetilde{T}^{\nu\mu} = 0. \tag{5.5}$$

This can be interpreted as an antisymmetric combination of a symmetric tensor [45], which we call $\Theta^{\mu\nu}$. A tensor form which reduces to the left-hand side of (5.5) upon antisymmetrization is

$$\Theta^{\mu\nu} = \widetilde{T}^{\mu\nu} + \frac{1}{8}\partial_{\sigma}\left[\overline{\psi}\{\gamma^{\sigma},\sigma^{\mu\nu}\}\psi - \overline{\psi}\{\gamma^{\mu},\sigma^{\sigma\nu}\}\psi - \overline{\psi}\{\gamma^{\nu},\sigma^{\sigma\mu}\}\psi\right]. \tag{5.6}$$

Using (5.5) to replace the left-most anti-commutator bilinear term, results in the manifestly symmetric combination

$$\Theta^{\mu\nu} = \frac{1}{2}(\widetilde{T}^{\mu\nu} + \widetilde{T}^{\nu\mu}) - \frac{1}{8}\partial_{\sigma}\left[\overline{\psi}\{\gamma^{\mu},\sigma^{\sigma\nu}\}\psi + \overline{\psi}\{\gamma^{\nu},\sigma^{\sigma\mu}\}\psi\right], \tag{5.7}$$

where the spin density [40], defined in this case as

$$S^{\mu\sigma\nu} = -\frac{1}{4}\left[\overline{\psi}\{\gamma^{\mu},\sigma^{\sigma\nu}\}\psi + \overline{\psi}\{\gamma^{\nu},\sigma^{\sigma\mu}\}\psi\right] = 0, \tag{5.8}$$

[1]For more details on mathematical conventions, refer to [26].

vanishes when taking into account the identity in the Dirac-Clifford algebra

$$\{\gamma^\mu, \sigma^{\sigma\nu}\} = 2\epsilon^{\sigma\nu\mu\rho}\gamma_5\gamma_\rho. \tag{5.9}$$

We therefore obtain the form of the Belinfante stress-energy tensor for a free Dirac particle

$$\Theta^{\mu\nu} = \frac{1}{2}(\widetilde{T}^{\mu\nu} + \widetilde{T}^{\nu\mu}) = -\frac{\mathrm{i}}{4}\left[\overline{\psi}\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\overline{\psi})\gamma^\mu\psi\right] - \frac{\mathrm{i}}{4}\left[\overline{\psi}\gamma^\nu(\partial^\mu\psi) - (\partial^\mu\overline{\psi})\gamma^\nu\psi\right], \tag{5.10}$$

in agreement with Goedecke [20]. Note that the Belinfante tensor is conserved

$$\partial_\mu\Theta^{\mu\nu} = 0, \tag{5.11}$$

and is equivalent to the Noether symmetry current of the Poincaré group.

5.1.2 Belinfante tensor in bilinear form

Our current objective is to rewrite (5.10) in terms of Fierz bilinears, which means we need to derive a Fierz identity that expresses the spinorial object $[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]$ in terms of bilinears. Therefore, we are led to search for Fierz expansions in which this term is likely to appear. One example is

$$\begin{aligned} j_\nu[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] &= \frac{\mathrm{i}}{3}(\partial_\mu j^\sigma)s_{\nu\sigma} - \frac{\mathrm{i}}{3}j^\sigma(\partial_\mu s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\omega)k_\nu - \frac{1}{3}\omega(\partial_\mu k_\nu) \\ &\quad + \frac{1}{3}\sigma[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] - \frac{\mathrm{i}}{3}{}^*s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] \\ &\quad - \frac{\mathrm{i}}{3}k^\sigma[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi]. \end{aligned} \tag{5.12}$$

There are at least three other bilinear products whose Fierz expansions produce the desired term, namely $j_\nu[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]$, $k_\nu(\partial_\mu\sigma)$ and $k_\nu(\partial_\mu\omega)$. Their respective expanded forms, along with a much more detailed derivation of the Fierz identity, is given in appendix E. Using these four identities, we can combine them to give

$$\begin{aligned} [\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] &= (\sigma\omega)^{-1}\Big(-\frac{\mathrm{i}}{2}(\partial_\mu j^\sigma)(\omega s_{\nu\sigma} + \sigma{}^*s_{\nu\sigma}) - k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] \\ &\quad + \frac{1}{2}(\partial_\mu k_\nu)(\sigma^2 + \omega^2) + j_\nu\{\omega[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] + \sigma[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\} \\ &\quad + \frac{\mathrm{i}}{2}(\sigma s_{\nu\sigma} + \omega{}^*s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\Big), \end{aligned} \tag{5.13}$$

which obviously still requires some more work. Using the Fierz identities derived in section 2.3

$$[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] = -(\sigma^2 - \omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\omega + \mathrm{i}m^\nu(\partial_\mu n_\nu)\sigma], \tag{5.14}$$

$$[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] = -(\sigma^2 - \omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\sigma + \mathrm{i}m^\nu(\partial_\mu n_\nu)\omega], \tag{5.15}$$

we can replace the spinor terms on the second line of (5.13) with bilinears, but this still leaves the spinor terms on the third line. After a straightforward, but tedious, set of Fierz manipulations, we obtain the desired identity

$$\begin{aligned}\frac{\mathrm{i}}{2}(\sigma s_{\nu\sigma}+\omega^* s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] &= -\frac{1}{2}\sigma\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi]\\ &+\frac{1}{4}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]-\frac{3}{8}(\sigma^2+\omega^2)(\partial_\mu k_\nu)\\ &+\frac{3}{8}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]-\frac{\mathrm{i}}{8}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma})+\frac{\mathrm{i}}{8}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})].\end{aligned} \tag{5.16}$$

Substituting into (5.13) and rearranging yields

$$\begin{aligned}[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi] &= (\sigma\omega)^{-1}\left\{-\frac{1}{2}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]\right.\\ &+\frac{1}{12}(\sigma^2+\omega^2)(\partial_\mu k_\nu)-\frac{5}{12}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]+\frac{\mathrm{i}}{12}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})]\\ &\left.-\frac{5\mathrm{i}}{12}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma})\right\}.\end{aligned} \tag{5.17}$$

This is entirely in terms of bilinears, but we would like to go further and eliminate the rank-2 terms, $s_{\mu\nu}$ and ${}^*s_{\mu\nu}$. Using the known Fierz identities for the replacement of these terms [27]

$$s^{\mu\nu}=(\sigma^2-\omega^2)^{-1}(\sigma\epsilon^{\mu\nu\rho\sigma}-\omega\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{5.18}$$

$${}^*s^{\mu\nu}=(\sigma^2-\omega^2)^{-1}(\omega\epsilon^{\mu\nu\rho\sigma}-\sigma\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{5.19}$$

where we define the partially antisymmetric object

$$\delta^{\mu\nu\rho\sigma}\equiv\mathrm{i}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}), \tag{5.20}$$

the rank-2 dependent terms in (5.17) become

$$\begin{aligned}\frac{\mathrm{i}}{12}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})] &= \frac{1}{12}(\sigma^2-\omega^2)^{-1}\{2\sigma\omega k_\nu[\omega(\partial_\mu\sigma)-\sigma(\partial_\mu\omega)]\\ &+2\mathrm{i}\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}j^\sigma(\partial_\mu j^\rho)k^\epsilon+j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)\}-\frac{1}{12}(\partial_\mu k_\nu)(\sigma^2+\omega^2),\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}-\frac{5\mathrm{i}}{12}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma}) &= (\sigma^2-\omega^2)^{-1}\left\{\frac{5}{12}j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)-\frac{5\mathrm{i}}{6}\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon\right.\\ &\left.-\frac{5}{12}k_\nu(\sigma^2+\omega^2)[\omega(\partial_\mu\omega)-\sigma(\partial_\mu\sigma)]\right\},\end{aligned} \tag{5.22}$$

giving us the final form of our identity

$$\begin{aligned}[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi] &= (\sigma^2-\omega^2)^{-1}\{k_\nu[\omega(\partial_\mu\sigma)-\sigma(\partial_\mu\omega)]-\mathrm{i}\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon\\ &-\mathrm{i}j_\nu m^\sigma(\partial_\mu n_\sigma)\}.\end{aligned} \tag{5.23}$$

Substituting into (5.10), we obtain the bilinear form of the Belinfante stress-energy tensor for a free Dirac particle

$$\begin{aligned}\Theta_{\mu\nu,\mathrm{D}} = \frac{1}{4}(\sigma^2 - \omega^2)^{-1}\{&-\mathrm{i}[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] \\ &- j^\rho k^\epsilon[\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma) + \epsilon_{\mu\sigma\rho\epsilon}(\partial_\nu j^\sigma)] - j_\mu m^\sigma(\partial_\nu n_\sigma) - j_\nu m^\sigma(\partial_\mu n_\sigma)\}. \quad (5.24)\end{aligned}$$

5.1.3 Maxwell-Dirac Belinfante tensor

The full Maxwell-Dirac stress-energy tensor is

$$\Theta_{\mu\nu,\mathrm{MD}} = \Theta_{\mu\nu,\mathrm{D}} + \Theta_{\mu\nu,\mathrm{int}} + \Theta_{\mu\nu,\mathrm{em}}, \quad (5.25)$$

where $\Theta_{\mu\nu,\mathrm{int}}$ and $\Theta_{\mu\nu,\mathrm{em}}$ are the interaction and Maxwell field contributions respectively. The Maxwell contribution has the well known form

$$\Theta_{\mu\nu,\mathrm{em}} = \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma, \quad (5.26)$$

consistent with our metric signature $(+---)$, and the interaction term is

$$\Theta_{\mu\nu,\mathrm{int}} = \frac{q}{2}(j_\mu A_\nu + j_\nu A_\mu), \quad (5.27)$$

where the electromagnetic vector potential A_μ can be replaced by the gauge independent analogue B_μ using the definition from section 2.3

$$A_\mu = B_\mu + \frac{1}{2q}(\sigma^2 - \omega^2)^{-1}m^\sigma(\partial_\mu n_\sigma). \quad (5.28)$$

The gauge dependent bilinear terms in (5.27) cancel out the corresponding terms in (5.24) exactly, so the full Maxwell-Dirac Belinfante stress-energy tensor is

$$\begin{aligned}\Theta_{\mu\nu,\mathrm{MD}} = &\frac{1}{4}(\sigma^2-\omega^2)^{-1}\{-\mathrm{i}[k_\mu(\omega\partial_\nu\sigma-\sigma\partial_\nu\omega)+k_\nu(\omega\partial_\mu\sigma-\sigma\partial_\mu\omega)]-j_\sigma k_\kappa[\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho)+\epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)]\} \\ &+ \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu) + \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma, \quad (5.29)\end{aligned}$$

which is manifestly symmetric and gauge independent.

5.2 Maxwell-Dirac stress-energy tensor via general relativity

5.2.1 Bilinear form of Dirac Lagrangian

We will now derive the bilinear form of the Maxwell-Dirac stress-energy tensor again, this time using a completely different method. The approach outlined in section 5.1 involved the use of Fierz identities to convert the spinorial form of the Belinfante stress-energy for a free Dirac particle (5.10) into bilinear form (5.24), to which the known tensor forms of the interaction and Maxwell contributions were added, yielding (5.29).

This time around, we convert the Lagrangian for an interacting Dirac particle

$$\mathscr{L} = \frac{\mathrm{i}}{2}\left[\overline{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma^\mu\psi\right] - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\psi A_\mu, \tag{5.30}$$

into bilinear form, then use the definition of the stress-energy tensor from general relativity to obtain our result, which should in principle agree with (5.29). Note that this method from general relativity was used directly on the spinorial form of the Lagrangian (5.3) by Goedecke [20], who then demonstrated its equivalence with the Belinfante stress-energy tensor (5.10). We are pursuing a similar equivalence demonstration, with our focus being on the bilinear formalism. The bilinearization of (5.30), is obtained by simply substituting the contracted form of (5.23), giving

$$\begin{aligned}[\overline{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma^\mu\psi] = (\sigma^2-\omega^2)^{-1}\{k^\mu[\omega(\partial_\mu\sigma) - \sigma(\partial_\mu\omega)] - \mathrm{i}\epsilon^{\mu\sigma\rho\epsilon}(\partial_\mu j_\sigma)j_\rho k_\epsilon \\ - \mathrm{i}j^\mu m^\sigma(\partial_\mu n_\sigma)\}.\end{aligned} \tag{5.31}$$

Applying the definitions $\sigma \equiv \overline{\psi}\psi$, $j^\mu \equiv \overline{\psi}\gamma^\mu\psi$, and using the definition of B_μ (5.28), we obtain

$$\mathscr{L} = \frac{1}{2}(\sigma^2-\omega^2)^{-1}\{\mathrm{i}k^\rho[\omega(\partial_\rho\sigma) - \sigma(\partial_\rho\omega)] + \epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau\} - m\sigma - qj^\rho B_\rho. \tag{5.32}$$

5.2.2 Variational form of the stress-energy tensor

The total action for the gravitational field in the presence of matter is [44]

$$S = \frac{S_\mathrm{H}}{16\pi G} + S_\mathrm{M}, \tag{5.33}$$

where S_M is the action for matter fields (mass-energy). S_H is the Hilbert action, defined as

$$S_\mathrm{H} = \int \mathrm{d}^4x\sqrt{-g}R, \tag{5.34}$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, and d^4x is the invariant volume element. The variation of the action with respect to an arbitrary tensor field $\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$ takes the general form

$$\delta S = \int \mathrm{d}^4x\frac{\delta S}{\delta\Phi}\delta\Phi, \tag{5.35}$$

with contraction over the indices implied. The term $\delta S/\delta\Phi$ is called the functional derivative of S with respect to the tensor field Φ. Of main interest in variational theory are tensors Φ_0 which extremize the action, so that $\delta S = 0$, and hence

$$\left.\frac{\delta S}{\delta\Phi}\right|_{\Phi_0} = 0. \tag{5.36}$$

Extremizing the variation of the Hilbert action (5.34) with respect to the inverse metric leads to Einstein's equations in vacuum

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{H}}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{5.37}$$

Likewise, extremizing the gravitational action in the presence of matter (5.33), so that

$$\delta S = \frac{\delta S_{\mathrm{H}}}{16\pi G} + \delta S_{\mathrm{M}} = 0, \tag{5.38}$$

and equating the corresponding functional derivatives, yields

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}} = 0. \tag{5.39}$$

Comparing with the well-known form of Einstein's equations in the presence of matter

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5.40}$$

we can see that the stress-energy tensor is of the general form

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}. \tag{5.41}$$

5.2.3 Variational Maxwell-Dirac stress-energy tensor

The variation of the electromagnetically interacting Dirac matter action, ignoring the contribution of the Maxwell field itself for now, is given by

$$\delta S = \int \mathrm{d}^4x \left(\delta\sqrt{-g}\mathscr{L} + \sqrt{-g}\delta\mathscr{L}\right), \tag{5.42}$$

where the Lagrangian is given by (5.32), but with appropriate modifications to make it covariant in curved space. Since the invariant volume element d^4x and $\sqrt{-g}$ are scalar densities of weight -1 and $+1$ respectively, we must arrange for the Lagrangian to be manifestly a scalar. Notice that (5.32) contains a term dependent on the Levi-Civita symbol with upstairs indices, which is of weight -1. This implies that we should make the replacement

$$\epsilon^{\rho\sigma\kappa\tau} \to \frac{1}{\sqrt{-g}}\epsilon^{\rho\sigma\kappa\tau}. \tag{5.43}$$

In order to deal with the bilinear four-vectors we must introduce the vierbein fields [48], which locally relate the curved metric to the flat one

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \tag{5.44}$$

where Greek and Latin indices label curved and flat spacetime components respectively. The gamma matrices are modified such that

$$\gamma_\mu = e_\mu{}^a\gamma_a; \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \tag{5.45}$$

so the bilinears are now

$$j_\mu = e_\mu{}^a j_a; \qquad j_a = \overline{\psi}\gamma_a\psi, \tag{5.46}$$

$$k_\mu = e_\mu{}^a k_a; \qquad k_a = \overline{\psi}\gamma_5\gamma_a\psi. \tag{5.47}$$

The variation of the square root of the negative metric determinant is [44]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{5.48}$$

Noting that $h = \sqrt{-g}$, where h is the vierbein determinant, we can use the variation of (5.44) to alternatively write this as

$$\delta h = -h e_\mu{}^a (\delta e^\mu{}_a), \tag{5.49}$$

implying the reciprocal variation

$$\delta(h^{-1}) = h^{-1} e_\mu{}^a (\delta e^\mu{}_a). \tag{5.50}$$

In curved space, the Levi-Civita term in (5.32) becomes

$$h^{-1}\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho e_\sigma{}^a) j_a j_\kappa k_\tau + h^{-1}\epsilon^{\rho\sigma\kappa\tau} e_\sigma{}^a (\partial_\rho j_a) j_\kappa k_\tau. \tag{5.51}$$

Introducing the covariant derivative causes the first term to vanish, due to the tetrad postulate [48]

$$\nabla_\mu e_\nu{}^a = 0. \tag{5.52}$$

Expanding out all of the vierbein fields in the second term, we find that

$$\epsilon^{\rho\sigma\kappa\tau} e_\sigma{}^a (\partial_\rho j_a) j_\kappa k_\tau = \epsilon^{abcd}(\partial_a j_b) j_c k_d, \tag{5.53}$$

which implies that for any curved coordinate components, this term is always equal to the flat spacetime version, so it is automatically covariant. We find that the covariant bilinear electromagnetically interacting Dirac matter Lagrangian has the form

$$\begin{aligned}\mathscr{L} = {}& \frac{1}{2}(\sigma^2 - \omega^2)^{-1}[\mathrm{i} g^{\rho\sigma} e_\sigma{}^a k_a (\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) + h^{-1}\epsilon^{\rho\sigma\kappa\tau} e_\sigma{}^a e_\kappa{}^b e_\tau{}^c (\partial_\rho j_a) j_b k_c] \\ & - m\sigma - q g^{\rho\sigma} e_\sigma{}^a j_a B_\rho.\end{aligned} \tag{5.54}$$

The variation with respect to deformation of the vierbein field is

$$\begin{aligned}\delta\mathscr{L} = {}& \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}\delta(g^{\rho\sigma} e_\sigma{}^a) k_a (\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) + \epsilon^{\rho\sigma\kappa\tau}\delta(h^{-1} e_\sigma{}^a e_\kappa{}^b e_\tau{}^c)(\partial_\rho j_a) j_b k_c\} \\ & - q\delta(g^{\rho\sigma} e_\sigma{}^a) j_a B_\rho.\end{aligned} \tag{5.55}$$

From the variation of (5.44), we find that

$$\delta(g^{\rho\sigma} e_\sigma{}^a) = 2(\delta e^{\rho a}) + (\delta e^\sigma{}_b) e^{\rho b} e_\sigma{}^a. \tag{5.56}$$

Using the fundamental vierbein property

$$e_\mu{}^a e^\mu{}_b = \delta^a_b, \tag{5.57}$$

$$e_\mu{}^a(\delta e^\mu{}_b) = -(\delta e_\mu{}^a)e^\mu{}_b, \tag{5.58}$$

we find that the first and last terms in (5.55) are

$$\mathrm{i}\delta(g^{\rho\sigma}e_\sigma{}^a)k_a(\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) = -\mathrm{i}k_\mu(\omega\partial^a\sigma - \sigma\partial^a\omega)(\delta e^\mu{}_a), \tag{5.59}$$

$$-q\delta(g^{\rho\sigma}e_\sigma{}^a)j_aB_\rho = qj_\mu B^a(\delta e^\mu{}_a). \tag{5.60}$$

Following a similar process, we find that the second variational term is

$$\begin{aligned}&\epsilon^{\rho\sigma\kappa\tau}\delta(h^{-1}e_\sigma{}^a e_\kappa{}^b e_\tau{}^c)(\partial_\rho j_a)j_b k_c \\ &\quad= h^{-1}[-e_\mu{}^a\epsilon^{\rho\sigma\kappa\tau}e_\sigma{}^d(\partial_\rho j_d)j_\kappa k_\tau + \epsilon^\rho{}_\mu{}^{\sigma\kappa}(\partial_\rho j^a)j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma{}^b(\partial_\rho j_b)j^a k_\kappa \\ &\qquad + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma{}^b(\partial_\rho j_b)j_\kappa k^a](\delta e^\mu{}_a).\end{aligned} \tag{5.61}$$

Gathering the deformed terms together, we can write the variation of the Lagrangian as

$$\begin{aligned}\delta\mathscr{L} = \Big(&\frac{1}{2}(\sigma^2-\omega^2)^{-1}\{-\mathrm{i}k_\mu(\omega\partial^a\sigma - \sigma\partial^a\omega) + h^{-1}[-e_\mu{}^a\epsilon^{\rho\sigma\kappa\tau}e_\sigma{}^d(\partial_\rho j_d)j_\kappa k_\tau \\ &+\epsilon^\rho{}_\mu{}^{\sigma\kappa}(\partial_\rho j^a)j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma{}^b(\partial_\rho j_b)j^a k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma{}^b(\partial_\rho j_b)j_\kappa k^a]\} + qj_\mu B^a\Big)(\delta e^\mu{}_a),\end{aligned} \tag{5.62}$$

with the associated action variation being

$$\begin{aligned}\delta S_{\mathrm{D}} = \int \mathrm{d}^4x\sqrt{-g}\,\Big(&-e_\mu{}^a\left\{\frac{1}{2}(\sigma^2-\omega^2)^{-1}[\mathrm{i}k^\rho(\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) + \epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau] - m\sigma - qj^\rho B_\rho\right\} \\ &+\frac{1}{2}(\sigma^2-\omega^2)^{-1}\{-\mathrm{i}k_\mu(\omega\partial^a\sigma - \sigma\partial^a\omega) + h^{-1}[-e_\mu{}^a\epsilon^{\rho\sigma\kappa\tau}e_\sigma{}^d(\partial_\rho j_d)j_\kappa k_\tau + \epsilon^\rho{}_\mu{}^{\sigma\kappa}(\partial_\rho j^a)j_\sigma k_\kappa \\ &+\epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma{}^b(\partial_\rho j_b)j^a k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma{}^b(\partial_\rho j_b)j_\kappa k^a]\} + qj_\mu B^a\Big)(\delta e^\mu{}_a).\end{aligned} \tag{5.63}$$

From the general form of the action variation (5.35), a relationship between (5.63) and the stress-energy tensor can be obtained [48]

$$\delta S_{\mathrm{D}} = \int \mathrm{d}^4x\sqrt{-g}\,u_\lambda{}^a\delta e^\lambda{}_a = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}, \tag{5.64}$$

which implies that

$$u_\mu{}^a = \frac{1}{2}(T_{\mu\lambda}e^{\lambda a} + T_{\lambda\mu}e^{\lambda a}). \tag{5.65}$$

Recognizing $T_{\mu\nu}$ as symmetric gives

$$T_{\mu\nu} = \frac{1}{2}(e_{\mu a}u_\nu{}^a + e_{\nu a}u_\mu{}^a). \tag{5.66}$$

Identifying the contents of the external parentheses in (5.63) with $u_\mu{}^a$, we obtain for the stress energy tensor

$$\begin{aligned}T_{\mu\nu} = &-\eta_{\mu\nu}\left\{\frac{1}{2}(\sigma^2-\omega^2)^{-1}[\mathrm{i}k^\rho(\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) + \epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau] - m\sigma - qj^\rho B_\rho\right\} \\ &+\frac{1}{4}(\sigma^2-\omega^2)^{-1}\{-\mathrm{i}[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] - 2\eta_{\mu\nu}\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^{\rho}{}_{\mu}{}^{\sigma\kappa}(\partial_\rho j_\nu)j_\sigma k_\kappa+\epsilon^{\rho\sigma}{}_{\mu}{}^{\kappa}(\partial_\rho j_\sigma)j_\nu k_\kappa+\epsilon^{\rho\sigma\kappa}{}_{\mu}(\partial_\rho j_\sigma)j_\kappa k_\nu+\epsilon^{\rho}{}_{\nu}{}^{\sigma\kappa}(\partial_\rho j_\mu)j_\sigma k_\kappa\\
&+\epsilon^{\rho\sigma}{}_{\nu}{}^{\kappa}(\partial_\rho j_\sigma)j_\mu k_\kappa+\epsilon^{\rho\sigma\kappa}{}_{\nu}(\partial_\rho j_\sigma)j_\kappa k_\mu\}+\frac{q}{2}(j_\mu B_\nu+j_\nu B_\mu),
\end{aligned}
\tag{5.67}
$$

where we have evaluated at flat spacetime. This is manifestly symmetric, but it requires some additional manipulation before it more closely resembles the Belinfante form (5.29). Consider the $U(1)$ gauge covariant Dirac equation and its Dirac conjugate

$$
\mathrm{i}\gamma^\sigma(\partial_\sigma\psi)-q\gamma^\sigma A_\sigma\psi-m\psi=0, \tag{5.68}
$$

$$
\mathrm{i}(\partial_\sigma\overline{\psi})\gamma^\sigma+q\overline{\psi}\gamma^\sigma A_\sigma+m\overline{\psi}=0. \tag{5.69}
$$

Left and right multiplying these equations by $\overline{\psi}$ and ψ respectively, then subtracting the second from the first and rearranging, gives

$$
m\sigma=\frac{\mathrm{i}}{2}\left[\overline{\psi}\gamma^\sigma(\partial_\sigma\psi)-(\partial_\sigma\overline{\psi})\gamma^\sigma\psi\right]-qj^\sigma A_\sigma \tag{5.70}
$$

Applying the Fierz identity (5.31) and the B_μ definition (5.28), this becomes

$$
-m\sigma=-\frac{1}{2}(\sigma^2-\omega^2)^{-1}[\mathrm{i}k^\rho(\omega\partial_\rho\sigma-\sigma\partial_\rho\omega)+\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau]+qj^\rho B_\rho, \tag{5.71}
$$

causing the $\eta_{\mu\nu}$ dependent term in (5.67) to vanish. Now consider the combinatorial identity[2]

$$
\begin{aligned}
&-\epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)j_\sigma k_\kappa-\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho)j_\sigma k_\kappa\\
&\quad=-2\eta_{\mu\nu}\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau+\epsilon^{\rho}{}_{\mu}{}^{\sigma\kappa}(\partial_\rho j_\nu)j_\sigma k_\kappa+\epsilon^{\rho\sigma}{}_{\mu}{}^{\kappa}(\partial_\rho j_\sigma)j_\nu k_\kappa+\epsilon^{\rho\sigma\kappa}{}_{\mu}(\partial_\rho j_\sigma)j_\kappa k_\nu\\
&\qquad+\epsilon^{\rho}{}_{\nu}{}^{\sigma\kappa}(\partial_\rho j_\mu)j_\sigma k_\kappa+\epsilon^{\rho\sigma}{}_{\nu}{}^{\kappa}(\partial_\rho j_\sigma)j_\mu k_\kappa+\epsilon^{\rho\sigma\kappa}{}_{\nu}(\partial_\rho j_\sigma)j_\kappa k_\mu,
\end{aligned}
\tag{5.72}
$$

which can be used to obtain the final form of the variational stress-energy tensor for Dirac matter

$$
\begin{aligned}
T_{\mu\nu,\mathrm{D}}=&\frac{1}{4}(\sigma^2-\omega^2)^{-1}\{-\mathrm{i}[k_\mu(\omega\partial_\nu\sigma-\sigma\partial_\nu\omega)+k_\nu(\omega\partial_\mu\sigma-\sigma\partial_\mu\omega)]-j_\sigma k_\kappa[\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho)+\epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)]\}\\
&+\frac{q}{2}(j_\mu B_\nu+j_\nu B_\mu).
\end{aligned}
\tag{5.73}
$$

Comparing with the Belinfante tensor (5.29), we find that they agree

$$
T_{\mu\nu,\mathrm{MD}}=\Theta_{\mu\nu,\mathrm{MD}}, \tag{5.74}
$$

when the gauge field stress-energy (5.26) is included on the left-hand side.

[2]This follows from the 5 term cyclic identity $V^\alpha\epsilon^{\rho\sigma\kappa\tau}+V^\tau\epsilon^{\alpha\rho\sigma\kappa}+\cdots=0$ which holds for the Levi-Civita tensor multiplied by any contravariant vector quantity. With the role of V^α played by the Kronecker δ^α_β (for fixed β), this yields (5.72) after contracting with $\eta_{\mu\alpha}\delta^\beta_\nu(\partial_\rho j_\sigma)j_\kappa k_\tau$, and rearranging indices appropriately.

5.3 Symmetry reduction of the Maxwell-Dirac stress-energy tensor

The Maxwell-Dirac equations in the bilinear formalism are in general, a very complicated set of self-coupled partial differential equations. An application of the present construction of the physical stress-energy tensor of the system in terms of bilinears, is therefore to provide a representation of the conserved rest mass of possible solutions (via the spatial integral of T_{00} for example).

For any meaningful solutions to be derived, it is natural to consider reduced forms under the imposition of special symmetries. The reduction of the bilinear form of the Maxwell-Dirac system under several examples of subgroups of the Poincaré group was discussed in chapter 3. We therefore choose one of the most important of these subgroups to work with here, namely $SO(3)$, which corresponds to spherical symmetry. In particular, we shall demonstrate how the bilinear form of the Maxwell-Dirac stress-energy tensor reduces, given the restrictions imposed by this subgroup. The treatment of other symmetry reductions, such as cylindrical symmetry and the $\widetilde{P}_{13,10}$ subgroup from [35], shall be left for future work.

Under spherical symmetry, scalar fields (σ, ω, etc.) have the generic form

$$\phi = f(t, r) \tag{5.75}$$

and vector fields (j^μ, k^μ, etc.) have the form

$$\Phi^\mu = \begin{pmatrix} f(t,r) \\ xg(t,r) \\ yg(t,r) \\ zg(t,r) \end{pmatrix}, \tag{5.76}$$

where the invariant

$$r = \sqrt{x^2 + y^2 + z^2} \tag{5.77}$$

is simply the spatial radius. We showed in section 4.1 that the spherically symmetric forms of our bilinear vector and axial vector fields are

$$j^\mu = \begin{pmatrix} j_a \\ xj_b \\ yj_b \\ zj_b \end{pmatrix}, \qquad k^\mu = \begin{pmatrix} rj_b \\ (x/r)j_a \\ (y/r)j_a \\ (z/r)j_a \end{pmatrix}, \qquad B^\mu = \begin{pmatrix} B_a \\ xB_b \\ yB_b \\ zB_b \end{pmatrix}, \tag{5.78}$$

where the vector potential functions are

$$B_a = \left[\pm\frac{\mathrm{i}}{2}(\sigma_r\omega - \sigma\omega_r) - m\sigma j_a\right]\left[q(\sigma^2 - \omega^2)\right]^{-1}, \tag{5.79}$$

$$B_b = \left[\mp\frac{\mathrm{i}}{2r}(\sigma_t\omega - \sigma\omega_t) - m\sigma j_b\right]\left[q(\sigma^2 - \omega^2)\right]^{-1}. \tag{5.80}$$

Here, we are using a condensed derivative notation $\partial_t\sigma \equiv \sigma_t$, and so on. Note that the effect of the symmetry reduction has in this case, reduced the components of the four-vectors to the set of coefficient functions j_a, j_b, σ and ω, which are themselves further constrained by higher-order nonlinear PDEs in the Maxwell-Dirac system. The coefficient functions k_a and k_b have been eliminated through the use of the Fierz identities

$$j_\mu j^\mu = -k_\mu k^\mu = \sigma^2 - \omega^2, \tag{5.81}$$

$$j_\mu k^\mu = 0. \tag{5.82}$$

It is straightforward to show that the Levi-Civita terms in the stress-energy vanish in this symmetry case

$$j_\sigma k_\kappa [\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho) + \epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)] = 0. \tag{5.83}$$

The form of the stress-energy tensor we are dealing with is therefore

$$\begin{aligned} T_{\mu\nu,\mathrm{MD}} = & -\frac{\mathrm{i}}{4}(\sigma^2-\omega^2)^{-1}[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] + \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu) \\ & + \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma. \end{aligned} \tag{5.84}$$

For the $SO(3)$ symmetry, the components of the electromagnetic field strength tensor are

$$F_{0i} = x_i F_a(t,r), \tag{5.85}$$

$$F_{ij} = \epsilon_{ijk}x^k F_b(t,r), \tag{5.86}$$

where the Maxwell coefficient functions are

$$\begin{aligned} F_a(t,r) = & (1/qr)(\sigma^2-\omega^2)^{-2}\{-2m[\sigma j_a(\sigma\sigma_r - \omega\omega_r) + r\sigma j_b(\sigma\sigma_t - \omega\omega_t)] \\ & \pm \mathrm{i}[\sigma\omega(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2) + (\sigma^2+\omega^2)(\sigma_t\omega_t - \sigma_r\omega_r)]\} \\ & + (1/qr)(\sigma^2-\omega^2)^{-1}[m(\sigma_r j_a + \sigma j_{a,r} + r\sigma_t j_b + r\sigma j_{b,t}) \\ & \pm (\mathrm{i}/2)(\sigma_{tt}\omega - \sigma\omega_{tt} - \sigma_{rr}\omega + \sigma\omega_{rr})], \end{aligned} \tag{5.87}$$

representing the electric field form, and

$$F_b(r) = \pm\frac{1}{2qr^3}, \tag{5.88}$$

representing the magnetic field form, which happens to be that of a monopole. Treating the $\mu = \nu = 0$, $\mu = 0$, $\nu = i$ and $\mu = i$, $\nu = j$ cases separately, we find that the respective components of (5.84) are

$$T_{00,\mathrm{MD}} = T_a + \mathcal{F}, \tag{5.89}$$

$$T_{0i,\mathrm{MD}} = \frac{x_i}{r}T_b, \tag{5.90}$$

$$T_{ij,\mathrm{MD}} = \frac{x_i x_j}{r^2}T_c + \delta_{ij}\mathcal{F}, \tag{5.91}$$

where the energy density of the Maxwell field is

$$\mathcal{F} = \frac{r^2(F_a^2 + F_b^2)}{2}, \tag{5.92}$$

and the other functions are defined as

$$T_a(t,r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{\mathrm{i}}{2} \left[j_a(\sigma_r \omega - \sigma \omega_r) - r j_b(\sigma_t \omega - \sigma \omega_t) \right] - m\sigma j_a^2 \right\}, \tag{5.93}$$

$$T_b(t,r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{\mathrm{i}}{2} \left[j_a(\sigma_t \omega - \sigma \omega_t) - r j_b(\sigma_r \omega - \sigma \omega_r) \right] - m\sigma j_a r j_b \right\}, \tag{5.94}$$

$$T_c(t,r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{\mathrm{i}}{2} \left[j_a(\sigma_r \omega - \sigma \omega_r) - r j_b(\sigma_t \omega - \sigma \omega_t) \right] - m\sigma r^2 j_b^2 \right\} - 2\mathcal{F}. \tag{5.95}$$

Chapter 6

Solutions of the Reduced Maxwell-Dirac Equations

6.1 Static, spherically symmetric reduction

We now extend the spherical symmetry reduction discussed in 4.1 to include the more restrictive requirement of time translation invariance. By imposing this extra symmetry, we are in effect considering a *static* system, where all of the coefficient functions of t and r are reduced to functions of r only. Of course, this symmetry carries the implicit assumption that there exists a Lorentz reference frame whereby the four-current j^μ is entirely described by the charge density $\rho(r) \equiv j_a(r)$. The current density fluxes given by the spatial components of the four-current, $x^i j_b(r)$, would intuitively be expected to vanish in a physically realistic scenario, which we shall see is the case.

6.1.1 Maxwell-Dirac ODE

From 4.1.4, we have the spherically symmetric Maxwell-Dirac equations (4.36) and (4.37), along with the Fierz identity (4.38), the continuity equation (4.39), and the partial conservation of axial current (4.40). Noting that all of the terms on the right-hand side of (4.37) contain time derivative dependent objects, it immediately follows that for the static case,

$$j_b = 0. \tag{6.1}$$

Applying this, and the vanishing time derivative condition, our system further reduces to the Fierz identity

$$j_a^2 = \sigma^2 - \omega^2, \tag{6.2}$$

the partial conservation of axial current

$$\frac{2}{r} j_a + j_{a,r} = \mp 2\mathrm{i} m\omega, \tag{6.3}$$

and the Maxwell-Dirac equation for j_a

$$q^2 j_a = -(\sigma^2-\omega^2)^{-3} 4(\sigma\sigma_r - \omega\omega_r)\{-2m\sigma j_a(\sigma\sigma_r - \omega\omega_r) \pm \mathrm{i}[-\sigma_r\omega_r(\sigma^2+\omega^2)$$

$$
\begin{aligned}
&+ \sigma\omega(\sigma_r^2 + \omega_r^2)]\} + (\sigma^2 - \omega^2)^{-2}\{-2m[2(\sigma_r j_a + \sigma j_{a,r} + \sigma j_a/r)(\sigma\sigma_r - \omega\omega_r) \\
&+ \sigma j_a(\sigma_r^2 + \sigma\sigma_{rr} - \omega_r^2 - \omega\omega_{rr})] \pm \mathrm{i}[-2\sigma_r\omega_r(\sigma\sigma_r + \omega\omega_r + \sigma^2/r + \omega^2/r) \\
&+ (\sigma_{rr}\omega - \sigma\omega_{rr})(\sigma\sigma_r - \omega\omega_r) + (\sigma\omega_r + \sigma_r\omega + 2\sigma\omega/r)(\sigma_r^2 + \omega_r^2) \\
&- (\sigma^2 + \omega^2)(\sigma_r\omega_{rr} + \sigma_{rr}\omega_r) + 2\sigma\omega(\sigma_r\sigma_{rr} + \omega_r\omega_{rr})]\} \\
&+ (\sigma^2 - \omega^2)^{-1}\{m(\sigma_{rr} j_a + 2\sigma_r j_{a,r} + \sigma j_{a,rr} + 2\sigma_r j_a/r + 2\sigma j_{a,r}/r) \\
&\pm \mathrm{i}[(1/2)(\sigma\omega_{rrr} + \sigma_r\omega_{rr} - \sigma_{rr}\omega_r - \sigma_{rrr}\omega) + (1/r)(\sigma\omega_{rr} - \sigma_{rr}\omega)]\}. \qquad (6.4)
\end{aligned}
$$

The continuity equation (4.39) becomes $0 = 0$. Using (6.2) and (6.3), we can eliminate σ and ω from (6.4) entirely, resulting in an ODE in terms of $j_a(r)$ only. Upon rearrangement, we get

$$
\sigma = \pm \frac{1}{mr}\left(m^2 r^2 j_a^2 - j_a^2 - r j_a j_{a,r} - \frac{r^2}{4} j_{a,r}^2\right)^{1/2}, \qquad (6.5)
$$

$$
\omega = \mp \frac{\mathrm{i}}{m}\left(\frac{j_a}{r} + \frac{j_{a,r}}{2}\right), \qquad (6.6)
$$

and using the computational aid of *Mathematica*, their derivatives up to third order are

$$
\omega_r = \mp \frac{\mathrm{i}}{m}\left(-\frac{j_a}{r^2} + \frac{j_{a,r}}{r} + \frac{j_{a,rr}}{2}\right), \qquad (6.7)
$$

$$
\omega_{rr} = \mp \frac{\mathrm{i}}{m}\left(\frac{2j_a}{r^3} - \frac{2j_{a,r}}{r^2} + \frac{j_{a,rr}}{r} + \frac{j_{a,rrr}}{2}\right), \qquad (6.8)
$$

$$
\omega_{rrr} = \mp \frac{\mathrm{i}}{m}\left(-\frac{6j_a}{r^4} + \frac{6j_{a,r}}{r^3} - \frac{3j_{a,rr}}{r^2} + \frac{j_{a,rrr}}{r} + \frac{j_{a,rrrr}}{2}\right), \qquad (6.9)
$$

$$
\begin{aligned}
\sigma_r = {} & \pm[4mr^2(m^2r^2j_a^2 - j_a^2 - rj_aj_{a,r} - r^2j_{a,r}^2/4)^{1/2}]^{-1} \\
& (4j_a^2 - 2rj_aj_{a,r} + 4m^2r^3j_aj_{a,r} - 2r^2j_{a,r}^2 - 2r^2j_aj_{a,rr} - r^3j_{a,r}j_{a,rr}), \qquad (6.10)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{rr} = {} & \pm[16mr^3(m^2r^2j_a^2 - j_a^2 - rj_aj_{a,r} - r^2j_{a,r}^2/4)^{3/2}]^{-1} \\
& (32j_a^4 - 48m^2r^2j_a^4 + 16rj_a^3j_{a,r} + 16m^2r^3j_a^3j_{a,r} - 24r^2j_a^2j_{a,r}^2 - 20r^3j_aj_{a,r}^3 \\
& - 4r^4j_{a,r}^4 - 4m^2r^6j_{a,r}^4 + 16r^2j_a^3j_{a,rr} - 16m^2r^4j_a^3j_{a,rr} + 16m^4r^6j_a^3j_{a,rr} \\
& + 24r^3j_a^2j_{a,r}j_{a,rr} - 24m^2r^5j_a^2j_{a,r}j_{a,rr} + 12r^4j_aj_{a,r}^2j_{a,rr} + 4m^2r^6j_aj_{a,r}^2j_{a,rr} \\
& + 2r^5j_{a,r}^3j_{a,rr} - 4m^2r^6j_a^2j_{a,rr}^2 + 8r^3j_a^3j_{a,rrr} - 8m^2r^5j_a^3j_{a,rrr} + 12r^4j_a^2j_{a,r}j_{a,rrr} \\
& - 4m^2r^6j_a^2j_{a,r}j_{a,rrr} + 6r^5j_aj_{a,r}^2j_{a,rrr} + r^6j_{a,r}^3j_{a,rrr}), \qquad (6.11)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{rrr} = {} & \pm[64mr^4(m^2r^2j_a^2 - j_a^2 - rj_aj_{a,r} - r^2j_{a,r}^2/4)^{5/2}]^{-1} \\
& (384j_a^6 - 960m^2r^2j_a^6 + 768m^4r^4j_a^6 + 576rj_a^5j_{a,r} - 480m^2r^3j_a^5j_{a,r} - 384m^4r^5j_a^5j_{a,r} \\
& + 768m^2r^4j_a^4j_{a,r}^2 - 480r^3j_a^3j_{a,r}^3 + 480m^2r^5j_a^3j_{a,r}^3 - 360r^4j_a^2j_{a,r}^4 + 192m^2r^6j_a^2j_{a,r}^4 \\
& - 108r^5j_aj_{a,r}^5 + 24m^2r^7j_aj_{a,r}^5 + 48m^4r^9j_aj_{a,r}^5 - 12r^6j_{a,r}^6 - 24m^2r^8j_{a,r}^6 \\
& + 192r^2j_a^5j_{a,rr} - 480m^2r^4j_a^5j_{a,rr} + 480r^3j_a^4j_{a,r}j_{a,rr} - 432m^2r^5j_a^4j_{a,r}j_{a,rr} \\
& + 480r^4j_a^3j_{a,r}^2j_{a,rr} - 432m^2r^6j_a^3j_{a,r}^2j_{a,rr} + 240r^5j_a^2j_{a,r}^3j_{a,rr} - 24m^2r^7j_a^2j_{a,r}^3j_{a,rr} \\
& - 144m^4r^9j_a^2j_{a,r}^3j_{a,rr} + 60r^6j_aj_{a,r}^4j_{a,rr} + 72m^2r^8j_aj_{a,r}^4j_{a,rr} + 6r^7j_{a,r}^5j_{a,rr} \\
& + 144m^2r^6j_a^4j_{a,rr}^2 - 24m^2r^7j_a^3j_{a,r}j_{a,rr}^2 + 96m^4r^9j_a^3j_{a,r}j_{a,rr}^2 - 24m^2r^8j_a^2j_{a,r}^2j_{a,rr}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 12m^2r^9 j_a j_{a,r}^3 j_{a,rr}^2 - 24m^2r^8 j_a^3 j_{a,rr}^3 - 12m^2r^9 j_a^2 j_{a,r} j_{a,rr}^3 - 64r^3 j_a^5 j_{a,rrr} \\
&+ 96m^2r^5 j_a^5 j_{a,rrr} - 96m^4r^7 j_a^5 j_{a,rrr} + 64m^6r^9 j_a^5 j_{a,rrr} - 160r^4 j_a^4 j_{a,r} j_{a,rrr} \\
&+ 224m^2r^6 j_a^4 j_{a,r} j_{a,rrr} - 160m^4r^8 j_a^4 j_{a,r} j_{a,rrr} - 160r^5 j_a^3 j_{a,r}^2 j_{a,rrr} \\
&+ 120m^2r^7 j_a^3 j_{a,r}^2 j_{a,rrr} + 16m^4r^9 j_a^3 j_{a,r}^2 j_{a,rrr} - 80r^6 j_a^2 j_{a,r}^3 j_{a,rrr} - 20r^7 j_a j_{a,r}^4 j_{a,rrr} \\
&- 8m^2r^9 j_a j_{a,r}^4 j_{a,rrr} - 2r^8 j_{a,r}^5 j_{a,rrr} + 48m^2r^7 j_a^4 j_{a,rr} j_{a,rrr} - 48m^4r^9 j_a^4 j_{a,rr} j_{a,rrr} \\
&+ 48m^2r^8 j_a^3 j_{a,r} j_{a,rr} j_{a,rrr} + 12m^2r^9 j_a^2 j_{a,r}^2 j_{a,rr} j_{a,rrr} - 32r^4 j_a^5 j_{a,rrrr} \\
&+ 64m^2r^6 j_a^5 j_{a,rrrr} - 32m^4r^8 j_a^5 j_{a,rrrr} - 80r^5 j_a^4 j_{a,r} j_{a,rrrr} + 96m^2r^7 j_a^4 j_{a,r} j_{a,rrrr} \\
&- 16m^4r^9 j_a^4 j_{a,r} j_{a,rrrr} - 80r^6 j_a^3 j_{a,r}^2 j_{a,rrrr} + 48m^2r^8 j_a^3 j_{a,r}^2 j_{a,rrrr} \\
&- 40r^7 j_a^2 j_{a,r}^3 j_{a,rrrr} + 8m^2r^9 j_a^2 j_{a,r}^3 j_{a,rrrr} - 10r^8 j_a j_{a,r}^4 j_{a,rrrr} - r^9 j_{a,r}^5 j_{a,rrrr}).
\end{aligned} \tag{6.12}
$$

At this point, we make a change in our bilinear conventions and replace the pure imaginary pseudoscalar fields ω with their real analogue $\varpi \equiv \overline{\psi}\mathrm{i}\gamma_5\psi$. The conversion formula is

$$\omega = -\mathrm{i}\varpi = -\mathrm{i}\overline{\psi}\mathrm{i}\gamma_5\psi, \tag{6.13}$$

which can be implemented in (6.4) by changing the sign of terms quadratic in ω, ω_r, etc., and having single ω terms absorb free i factors without change of sign. We obtain the manifestly real expression

$$
\begin{aligned}
q^2 j_a = &-(\sigma^2+\varpi^2)^{-3}4(\sigma\sigma_r+\varpi\varpi_r)\{-2m\sigma j_a(\sigma\sigma_r+\varpi\varpi_r) \pm [-\sigma_r\varpi_r(\sigma^2-\varpi^2) \\
&+ \sigma\varpi(\sigma_r^2-\varpi_r^2)]\} + (\sigma^2+\varpi^2)^{-2}\{-2m[2(\sigma_r j_a+\sigma j_{a,r}+\sigma j_a/r)(\sigma\sigma_r+\varpi\varpi_r) \\
&+ \sigma j_a(\sigma_r^2+\sigma\sigma_{rr}+\varpi_r^2+\varpi\varpi_{rr})] \pm [-2\sigma_r\varpi_r(\sigma\sigma_r-\varpi\varpi_r+\sigma^2/r-\varpi^2/r) \\
&+ (\sigma_{rr}\varpi-\sigma\varpi_{rr})(\sigma\sigma_r+\varpi\varpi_r) + (\sigma\varpi_r+\sigma_r\varpi+2\sigma\varpi/r)(\sigma_r^2-\varpi_r^2) \\
&- (\sigma^2-\varpi^2)(\sigma_r\varpi_{rr}+\sigma_{rr}\varpi_r) + 2\sigma\varpi(\sigma_r\sigma_{rr}-\varpi_r\varpi_{rr})]\} \\
&+ (\sigma^2+\varpi^2)^{-1}\{m(\sigma_{rr}j_a+2\sigma_r j_{a,r}+\sigma j_{a,rr}+2\sigma_r j_a/r+2\sigma j_{a,r}/r) \\
&\pm [(1/2)(\sigma\varpi_{rrr}+\sigma_r\varpi_{rr}-\sigma_{rr}\varpi_r-\sigma_{rrr}\varpi) + (1/r)(\sigma\varpi_{rr}-\sigma_{rr}\varpi)]\},
\end{aligned} \tag{6.14}
$$

where we now have

$$\varpi = \pm\frac{1}{m}\left(\frac{j_a}{r}+\frac{j_{a,r}}{2}\right), \tag{6.15}$$

$$\varpi_r = \pm\frac{1}{m}\left(-\frac{j_a}{r^2}+\frac{j_{a,r}}{r}+\frac{j_{a,rr}}{2}\right), \tag{6.16}$$

$$\varpi_{rr} = \pm\frac{1}{m}\left(\frac{2j_a}{r^3}-\frac{2j_{a,r}}{r^2}+\frac{j_{a,rr}}{r}+\frac{j_{a,rrr}}{2}\right), \tag{6.17}$$

$$\varpi_{rrr} = \pm\frac{1}{m}\left(-\frac{6j_a}{r^4}+\frac{6j_{a,r}}{r^3}-\frac{3j_{a,rr}}{r^2}+\frac{j_{a,rrr}}{r}+\frac{j_{a,rrrr}}{2}\right). \tag{6.18}$$

Eliminating σ, ϖ, and all of their derivatives from (6.14) using *Mathematica*'s `ReplaceAll` function, and simplifying the resulting expression with the command `Factor[Expand[%]]`, we obtain an ODE in terms of r, j_a and its derivatives up to fourth order

$$\pm q^2 j_a = [64 r j_a^3 (m^2r^2j_a^2 - j_a^2 - r j_a j_{a,r} - r^2 j_{a,r}^2/4)^{5/2}]^{-1}$$

$$
\begin{aligned}
&(32m^2 j_a^8 - 128m^4 r^2 j_a^8 + 128m^2 r j_a^7 j_{a,r} - 16 j_a^6 j_{a,r}^2 - 96m^2 r^2 j_a^6 j_{a,r}^2 + 192 r j_a^5 j_{a,r}^3 \\
&- 240m^2 r^3 j_a^5 j_{a,r}^3 + 32m^4 r^5 j_a^5 j_{a,r}^3 + 108 r^2 j_a^4 j_{a,r}^4 + 136m^2 r^4 j_a^4 j_{a,r}^4 - 144m^4 r^6 j_a^4 j_{a,r}^4 \\
&- 124 r^3 j_a^3 j_{a,r}^5 + 216m^2 r^5 j_a^3 j_{a,r}^5 - 120 r^4 j_a^2 j_{a,r}^6 + 40m^2 r^6 j_a^2 j_{a,r}^6 - 36 r^5 j_a j_{a,r}^7 \\
&- 4r^6 j_{a,r}^8 + 144m^2 r^2 j_a^7 j_{a,rr} - 256 r j_a^6 j_{a,r} j_{a,rr} + 288m^2 r^3 j_a^6 j_{a,r} j_{a,rr} \\
&- 32m^4 r^5 j_a^6 j_{a,r} j_{a,rr} - 8r^2 j_a^5 j_{a,r}^2 j_{a,rr} - 344m^2 r^4 j_a^5 j_{a,r}^2 j_{a,rr} + 272m^4 r^6 j_a^5 j_{a,r}^2 j_{a,rr} \\
&+ 312 r^3 j_a^4 j_{a,r}^3 j_{a,rr} - 392m^2 r^5 j_a^4 j_{a,r}^3 j_{a,rr} + 214 r^4 j_a^3 j_{a,r}^4 j_{a,rr} - 56m^2 r^6 j_a^3 j_{a,r}^4 j_{a,rr} \\
&+ 56 r^5 j_a^2 j_{a,r}^5 j_{a,rr} + 6r^6 j_a j_{a,r}^6 j_{a,rr} - 112 r^2 j_a^6 j_{a,rr}^2 + 104m^2 r^4 j_a^6 j_{a,rr}^2 \\
&- 64m^4 r^6 j_a^6 j_{a,rr}^2 - 96 r^3 j_a^5 j_{a,r} j_{a,rr}^2 + 48m^2 r^5 j_a^5 j_{a,r} j_{a,rr}^2 - 12 r^4 j_a^4 j_{a,r}^2 j_{a,rr}^2 \\
&- 20m^2 r^6 j_a^4 j_{a,r}^2 j_{a,rr}^2 + 4r^5 j_a^3 j_{a,r}^3 j_{a,rr}^2 + 8r^4 j_a^5 j_{a,rr}^3 + 4m^2 r^6 j_a^5 j_{a,rr}^3 \\
&+ 8r^5 j_a^4 j_{a,r} j_{a,rr}^3 + 2r^6 j_a^3 j_{a,r}^2 j_{a,rr}^3 + 48 r j_a^7 j_{a,rrr} - 48m^2 r^3 j_a^7 j_{a,rrr} - 8r^2 j_a^6 j_{a,r} j_{a,rrr} \\
&+ 136m^2 r^4 j_a^6 j_{a,r} j_{a,rrr} - 80m^4 r^6 j_a^6 j_{a,r} j_{a,rrr} - 100 r^3 j_a^5 j_{a,r}^2 j_{a,rrr} \\
&+ 136m^2 r^5 j_a^5 j_{a,r}^2 j_{a,rrr} - 78 r^4 j_a^4 j_{a,r}^3 j_{a,rrr} + 28m^2 r^6 j_a^4 j_{a,r}^3 j_{a,rrr} - 22 r^5 j_a^3 j_{a,r}^4 j_{a,rrr} \\
&- 2r^6 j_a^2 j_{a,r}^5 j_{a,rrr} - 24 r^3 j_a^6 j_{a,rr} j_{a,rrr} + 24m^2 r^5 j_a^6 j_{a,rr} j_{a,rrr} - 36 r^4 j_a^5 j_{a,r} j_{a,rr} j_{a,rrr} \\
&+ 12m^2 r^6 j_a^5 j_{a,r} j_{a,rr} j_{a,rrr} - 18 r^5 j_a^4 j_{a,r}^2 j_{a,rr} j_{a,rrr} - 3r^6 j_a^3 j_{a,r}^3 j_{a,rr} j_{a,rrr} \\
&+ 16 r^2 j_a^7 j_{a,rrrr} - 32m^2 r^4 j_a^7 j_{a,rrrr} + 16m^4 r^6 j_a^7 j_{a,rrrr} + 32 r^3 j_a^6 j_{a,r} j_{a,rrrr} \\
&- 32m^2 r^5 j_a^6 j_{a,r} j_{a,rrrr} + 24 r^4 j_a^5 j_{a,r}^2 j_{a,rrrr} - 8m^2 r^6 j_a^5 j_{a,r}^2 j_{a,rrrr} + 8r^5 j_a^4 j_{a,r}^3 j_{a,rrrr} \\
&+ r^6 j_a^3 j_{a,r}^4 j_{a,rrrr}),
\end{aligned} \tag{6.19}
$$

the $j_a(r)$ solutions of which are the Maxwell-Dirac solutions we are interested in. Note that the sign ambiguity comes from the σ terms. Rearranging this expression yields an implicit expression of the form

$$f(r, j_a, j_{a,r}, j_{a,rr}, j_{a,rrr}, j_{a,rrrr}; m, q) = 0, \tag{6.20}$$

a non-linear, fourth order ODE with dimensional parameters m and q.

6.1.2 Natural L-H units and non-dimensionalization

In natural units, commonly used in high-energy physics [1],

$$[M] = [L]^{-1} = [T]^{-1}, \tag{6.21}$$

so r has units of both length and reciprocal mass. Additionally, charge q is a dimensionless quantity, which can be deduced from the form of the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon\hbar c} \approx \frac{1}{137}, \tag{6.22}$$

where e is the elementary charge. We are working in natural, Lorentz-Heaviside (L-H) units, where

$$\hbar = c = \epsilon_0 = 1, \tag{6.23}$$

so we can write e as

$$e = \sqrt{4\pi\alpha} \approx 0.302815, \tag{6.24}$$

which is obviously dimensionless. We find that in order for all of the terms in (6.19) to have the same dimensions, j_a must have dimensionality

$$[j_a] = [L]^{-3}, \tag{6.25}$$

which can be interpreted as a number density. We proceed to non-dimensionalize the system by defining dimensionless parameters

$$\chi = mr, \tag{6.26}$$
$$\bar{j}_a = q^2 m^{-3} j_a, \tag{6.27}$$
$$\bar{j}_{a,\chi} = q^2 m^{-4} j_{a,r}, \tag{6.28}$$

and so on, up to fourth-order derivatives. For the case where the Dirac fermion field corresponds to the electron or positron, $q = e$, so

$$q^2 = 4\pi\alpha, \tag{6.29}$$

which is essentially a dimensionless scaling factor proportional to the fine structure constant. This particular choice of non-dimensionalization allows $m^{26}q^{-16}$ to factored from (6.20) entirely, and we are left with a dimensionless ODE of implicit form

$$f(\chi, \bar{j}_a, \bar{j}_{a,\chi}, \bar{j}_{a,\chi\chi}, \bar{j}_{a,\chi\chi\chi}, \bar{j}_{a,\chi\chi\chi\chi}) = 0, \tag{6.30}$$

which is linear in the highest order derivative $\bar{j}_{a,\chi\chi\chi\chi}$, and contains no free parameters. Explicitly, our dimensionless ODE is

$$\begin{aligned}
0 = &\pm 4\chi\bar{j}_a^{4}[(\chi^2-1)\bar{j}_a^{2} - \chi\bar{j}_a\bar{j}_{a,\chi} - (\chi^2/4)\bar{j}_{a,\chi}^{2}]^{5/2} \\
&- 32\bar{j}_a^{8} + 128\chi^2\bar{j}_a^{8} - 128\chi\bar{j}_a^{7}\bar{j}_{a,\chi} + 16\bar{j}_a^{6}\bar{j}_{a,\chi}^{2} + 96\chi^2\bar{j}_a^{6}\bar{j}_{a,\chi}^{2} - 192\chi\bar{j}_a^{5}\bar{j}_{a,\chi}^{3} \\
&+ 240\chi^3\bar{j}_a^{5}\bar{j}_{a,\chi}^{3} - 32\chi^5\bar{j}_a^{5}\bar{j}_{a,\chi}^{3} - 108\chi^2\bar{j}_a^{4}\bar{j}_{a,\chi}^{4} - 136\chi^4\bar{j}_a^{4}\bar{j}_{a,\chi}^{4} + 144\chi^6\bar{j}_a^{4}\bar{j}_{a,\chi}^{4} \\
&+ 124\chi^3\bar{j}_a^{3}\bar{j}_{a,\chi}^{5} - 216\chi^5\bar{j}_a^{3}\bar{j}_{a,\chi}^{5} + 120\chi^4\bar{j}_a^{2}\bar{j}_{a,\chi}^{6} - 40\chi^6\bar{j}_a^{2}\bar{j}_{a,\chi}^{6} + 36\chi^5\bar{j}_a\bar{j}_{a,\chi}^{7} \\
&+ 4\chi^6\bar{j}_{a,\chi}^{8} - 144\chi^2\bar{j}_a^{7}\bar{j}_{a,\chi\chi} + 256\chi\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi} - 288\chi^3\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi} \\
&+ 32\chi^5\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi} + 8\chi^2\bar{j}_a^{5}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi} + 344\chi^4\bar{j}_a^{5}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi} - 272\chi^6\bar{j}_a^{5}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi} \\
&- 312\chi^3\bar{j}_a^{4}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi} + 392\chi^5\bar{j}_a^{4}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi} - 214\chi^4\bar{j}_a^{3}\bar{j}_{a,\chi}^{4}\bar{j}_{a,\chi\chi} + 56\chi^6\bar{j}_a^{3}\bar{j}_{a,\chi}^{4}\bar{j}_{a,\chi\chi} \\
&- 56\chi^5\bar{j}_a^{2}\bar{j}_{a,\chi}^{5}\bar{j}_{a,\chi\chi} - 6\chi^6\bar{j}_a\bar{j}_{a,\chi}^{6}\bar{j}_{a,\chi\chi} + 112\chi^2\bar{j}_a^{6}\bar{j}_{a,\chi\chi}^{2} - 104\chi^4\bar{j}_a^{6}\bar{j}_{a,\chi\chi}^{2} \\
&+ 64\chi^6\bar{j}_a^{6}\bar{j}_{a,\chi\chi}^{2} + 96\chi^3\bar{j}_a^{5}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi}^{2} - 48\chi^5\bar{j}_a^{5}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi}^{2} + 12\chi^4\bar{j}_a^{4}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi}^{2} \\
&+ 20\chi^6\bar{j}_a^{4}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi}^{2} - 4\chi^5\bar{j}_a^{3}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi}^{2} - 8\chi^4\bar{j}_a^{5}\bar{j}_{a,\chi\chi}^{3} - 4\chi^6\bar{j}_a^{5}\bar{j}_{a,\chi\chi}^{3} - 8\chi^5\bar{j}_a^{4}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi}^{3} \\
&- 2\chi^6\bar{j}_a^{3}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi}^{3} - 48\chi\bar{j}_a^{7}\bar{j}_{a,\chi\chi\chi} + 48\chi^3\bar{j}_a^{7}\bar{j}_{a,\chi\chi\chi} + 8\chi^2\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi\chi} \\
&- 136\chi^4\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi\chi} + 80\chi^6\bar{j}_a^{6}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi\chi} + 100\chi^3\bar{j}_a^{5}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi\chi} \\
&- 136\chi^5\bar{j}_a^{5}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi\chi} + 78\chi^4\bar{j}_a^{4}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi\chi} - 28\chi^6\bar{j}_a^{4}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi\chi} + 22\chi^5\bar{j}_a^{3}\bar{j}_{a,\chi}^{4}\bar{j}_{a,\chi\chi\chi} \\
&+ 2\chi^6\bar{j}_a^{2}\bar{j}_{a,\chi}^{5}\bar{j}_{a,\chi\chi\chi} + 24\chi^3\bar{j}_a^{6}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} - 24\chi^5\bar{j}_a^{6}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} \\
&+ 36\chi^4\bar{j}_a^{5}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} - 12\chi^6\bar{j}_a^{5}\bar{j}_{a,\chi}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} + 18\chi^5\bar{j}_a^{4}\bar{j}_{a,\chi}^{2}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} \\
&+ 3\chi^6\bar{j}_a^{3}\bar{j}_{a,\chi}^{3}\bar{j}_{a,\chi\chi}\bar{j}_{a,\chi\chi\chi} - 16\chi^2\bar{j}_a^{7}\bar{j}_{a,\chi\chi\chi\chi} + 32\chi^4\bar{j}_a^{7}\bar{j}_{a,\chi\chi\chi\chi} - 16\chi^6\bar{j}_a^{7}\bar{j}_{a,\chi\chi\chi\chi}
\end{aligned}$$

$$
\begin{aligned}
&- 32\chi^3\bar{\jmath}_a^6\bar{\jmath}_{a,\chi}\bar{\jmath}_{a,\chi\chi\chi\chi} + 32\chi^5\bar{\jmath}_a^6\bar{\jmath}_{a,\chi}\bar{\jmath}_{a,\chi\chi\chi\chi} - 24\chi^4\bar{\jmath}_a^5\bar{\jmath}_{a,\chi}^2\bar{\jmath}_{a,\chi\chi\chi\chi} \\
&+ 8\chi^6\bar{\jmath}_a^5\bar{\jmath}_{a,\chi}^2\bar{\jmath}_{a,\chi\chi\chi\chi} - 8\chi^5\bar{\jmath}_a^4\bar{\jmath}_{a,\chi}^3\bar{\jmath}_{a,\chi\chi\chi\chi} - \chi^6\bar{\jmath}_a^3\bar{\jmath}_{a,\chi}^4\bar{\jmath}_{a,\chi\chi\chi\chi},
\end{aligned}
\tag{6.31}
$$

6.1.3 An exact solution

Notice that in the above ODE, the presence of the square root term provides us with the additional constraint on the argument

$$
(\chi^2 - 1)\bar{\jmath}_a^2 - \chi\bar{\jmath}_a\bar{\jmath}_{a,\chi} - (\chi^2/4)\bar{\jmath}_{a,\chi}^2 \geq 0, \tag{6.32}
$$

since we are restricting ourselves to real solutions. We shall focus on the special case where we have the equality

$$
(\chi^2 - 1)\bar{\jmath}_a^2 - \chi\bar{\jmath}_a\bar{\jmath}_{a,\chi} - (\chi^2/4)\bar{\jmath}_{a,\chi}^2 = 0, \tag{6.33}
$$

and solve for $\bar{\jmath}_{a,\chi}$. From the quadratic formula, we have

$$
\bar{\jmath}_{a,\chi} = \frac{\chi\bar{\jmath}_a \pm \sqrt{\chi^2\bar{\jmath}_a^2 - 4(-\chi^2/4)[(\chi^2-1)\bar{\jmath}_a^2]}}{2(-\chi^2/4)}, \tag{6.34}
$$

which simplifies to give

$$
\bar{\jmath}_{a,\chi} = -2(\chi^{-1} \pm 1)\bar{\jmath}_a. \tag{6.35}
$$

Noting that this is a separable ODE, we can rearrange to obtain

$$
\frac{\mathrm{d}\bar{\jmath}_a}{\bar{\jmath}_a} = -2(\chi^{-1} \pm 1)\mathrm{d}\chi. \tag{6.36}
$$

Integrating, then exponentiating yields the solution

$$
\bar{\jmath}_a = A\chi^{-2}\mathrm{e}^{\mp 2\chi}, \tag{6.37}
$$

where A is an arbitrary constant. Not only does this $\bar{\jmath}_a$ form solve (6.33), but it also happens to be a solution of the full static spherically symmetric Maxwell-Dirac ODE (6.31), which can be checked explicitly using *Mathematica*'s `ReplaceAll` function. The square root term, and everything to the right of it, vanishes independently. The special nature of this solution can be understood by observing that every term on the right hand side of (6.14) is multiplied by σ, or one of its derivatives, and that imposing the condition (6.33) is equivalent to setting $\sigma = 0$ in (6.5).

6.1.4 Linearization about the exact solution

Now consider the case where we add a perturbing function to the exact solution, so that

$$
\begin{aligned}
\bar{\jmath}_a(\chi) &= J_0(\chi) + \sum_{n=1}^{\infty} \epsilon^n J_n(\chi) \\
&= J_0(\chi) + \epsilon J_1(\chi) + \mathrm{O}(\epsilon^2),
\end{aligned}
\tag{6.38}
$$

where

$$J_0(\chi) = A\chi^{-2}\mathrm{e}^{-2\chi} \tag{6.39}$$

is the exponentially decreasing exact solution, and the small-ϵ dependent part is the perturbation. By linearizing, we shall only retain terms of the lowest order in ϵ. Substituting (6.38) into (6.31), we find that the lowest order terms are proportional to $\epsilon^{5/2}$, and we obtain the equation

$$4A^6\chi^{-9}\mathrm{e}^{-12\chi}(2J_1 + 2\chi J_1 + \chi J_1')^2\sqrt{A\chi^{-1}\mathrm{e}^{-2\chi}(2J_1 + 2\chi J_1 + \chi J_1')}\epsilon^{5/2} + O(\epsilon^3) = 0, \tag{6.40}$$

which when truncated to the lowest order and the generally non-zero terms discarded, becomes the simple expression

$$2J_1 + 2\chi J_1 + \chi J_1' = 0, \tag{6.41}$$

or alternatively,

$$J_1' = -2(\chi^{-1} + 1)J_1. \tag{6.42}$$

This has exactly the same form as the ODE (6.35) that led to our exact solution (6.37), so we can immediately say

$$J_1 = A'\chi^{-2}\mathrm{e}^{-2\chi}. \tag{6.43}$$

Therefore, our first order perturbed solution is

$$\bar{\jmath}_a = A\chi^{-2}\mathrm{e}^{-2\chi} + \epsilon A'\chi^{-2}\mathrm{e}^{-2\chi} = A''\chi^{-2}\mathrm{e}^{-2\chi}, \tag{6.44}$$

which is just our original exact solution, but with a slightly different arbitrary coefficient, $A'' = A + \epsilon A'$. It may be the case that our exact solution is analogous to the so-called "singular solutions" obtained from the first integral of the simple harmonic motion ODE [25]. This would imply that the exact solution (6.37) is an envelope of a family of solutions, for a given A. Whatever the case, solutions of the form (6.37) do not correspond to physically realistic situations in either sign case, due to the large singularity at the origin caused by the χ^{-2} contribution.

6.1.5 Equilibrium points

We will now determine the equilibrium points of the system, which are points in the solution phase space where all of the derivatives vanish. Observing the terms in (6.31), we can see that only the first term inside the square root term, as well as the first two terms following this, are non-zero in this case. The ODE corresponding to the equilibria is therefore

$$\pm 4\chi\bar{\jmath}_{a,e}^{4}[(\chi^2 - 1)\bar{\jmath}_{a,e}{}^2]^{5/2} - 32\bar{\jmath}_{a,e}^{8} + 128\chi^2\bar{\jmath}_{a,e}^{8} = 0, \tag{6.45}$$

which factorizes to

$$\bar{\jmath}_{a,e}^{8}[8(4\chi^2 - 1) \pm \chi(\chi^2 - 1)^{5/2}\bar{\jmath}_{a,e}] = 0, \tag{6.46}$$

where $\bar{j}_{a,e}$ are the solutions at the equilibrium points. We can easily see that there are three solutions to this equation

$$\bar{j}_{a,e} = 0, \tag{6.47}$$

$$\bar{j}_{a,e} = \pm\frac{8(1-4\chi^2)}{\chi(\chi^2-1)^{5/2}}. \tag{6.48}$$

Looking at the form of the second solution, we can see that there are singularities at $\chi = 0$ and $\chi = \pm 1$. Furthermore, for $0 < \chi < 1$, the square root argument $\chi^2 - 1 < 0$, causing the second two solutions to be pure imaginary in this interval. Due to this unphysical behaviour, we shall restrict further investigation to the $\bar{j}_{a,e} = 0$ equilibrium point.

6.1.6 Weakly non-linear ODE and spectral method

Consider a first order perturbation about the equilibrium point $\bar{j}_{a,e} = 0$,

$$\bar{j}_a = \epsilon J + \mathrm{O}(\epsilon^2), \tag{6.49}$$

which is equivalent to the *direct linearization* of $\bar{j}_a$. Substituting this perturbative form into (6.31), we find that the square root term is proportional to ϵ^9, and the large number of terms following this are all proportional to ϵ^8. Therefore, retaining only the lowest order terms in ϵ, the square root term disappears, and we are left with the *weakly non-linear* ODE

$$\begin{aligned}
0 = {} & (32J^8 - 16J^6{J_\chi}^2) + (128J^7J_\chi + 192J^5{J_\chi}^3 - 256J^6J_\chi J_{\chi\chi} + 48\chi J^7 J_{\chi\chi\chi})\chi \\
& + (-128J^8 - 96J^6{J_\chi}^2 + 108J^4{J_\chi}^4 + 144J^7J_{\chi\chi} - 8\chi^2J^5{J_\chi}^2J_{\chi\chi} - 112J^6{J_{\chi\chi}}^2 \\
& - 8J^6J_\chi J_{\chi\chi\chi} + 16\chi^2J^7J_{\chi\chi\chi\chi})\chi^2 + (-240J^5{J_\chi}^3 - 124J^3{J_\chi}^5 + 288J^6J_\chi J_{\chi\chi} \\
& + 312J^4{J_\chi}^3J_{\chi\chi} - 96J^5J_\chi{J_{\chi\chi}}^2 - 48J^7J_{\chi\chi\chi} - 100J^5{J_\chi}^2J_{\chi\chi\chi} - 24J^6J_{\chi\chi}J_{\chi\chi\chi} \\
& + 32J^6J_\chi J_{\chi\chi\chi\chi})\chi^3 + (136J^4{J_\chi}^4 - 120J^2{J_\chi}^6 - 344J^5{J_\chi}^2J_{\chi\chi} + 214J^3{J_\chi}^4J_{\chi\chi} \\
& + 104J^6{J_{\chi\chi}}^2 - 12J^4{J_\chi}^2{J_{\chi\chi}}^2 + 8J^5{J_{\chi\chi}}^3 + 136J^6J_\chi J_{\chi\chi\chi} - 78J^4{J_\chi}^3J_{\chi\chi\chi} \\
& - 36J^5J_\chi J_{\chi\chi}J_{\chi\chi\chi} - 32J^7J_{\chi\chi\chi\chi} + 24J^5{J_\chi}^2J_{\chi\chi\chi\chi})\chi^4 + (32J^5{J_\chi}^3 + 216J^3{J_\chi}^5 \\
& - 36J{J_\chi}^7 - 32J^6J_\chi J_{\chi\chi} - 392J^4{J_\chi}^3J_{\chi\chi} + 56J^2{J_\chi}^5J_{\chi\chi} + 48J^5J_\chi{J_{\chi\chi}}^2 \\
& + 4J^3{J_\chi}^3{J_{\chi\chi}}^2 + 8J^4J_\chi{J_{\chi\chi}}^3 + 136J^5{J_\chi}^2J_{\chi\chi\chi} - 22J^3{J_\chi}^4J_{\chi\chi\chi} + 24J^6J_{\chi\chi}J_{\chi\chi\chi} \\
& - 18J^4{J_\chi}^2J_{\chi\chi}J_{\chi\chi\chi} - 32J^6J_\chi J_{\chi\chi\chi\chi} + 8J^4{J_\chi}^3J_{\chi\chi\chi\chi})\chi^5 + (-144J^4{J_\chi}^4 \\
& + 40J^2{J_\chi}^6 - 4{J_\chi}^8 + 272J^5{J_\chi}^2J_{\chi\chi} - 56J^3{J_\chi}^4J_{\chi\chi} + 6J{J_\chi}^6J_{\chi\chi} - 64J^6{J_{\chi\chi}}^2 \\
& - 20J^4{J_\chi}^2{J_{\chi\chi}}^2 + 4J^5{J_{\chi\chi}}^3 + 2J^3{J_\chi}^2{J_{\chi\chi}}^3 - 80J^6J_\chi J_{\chi\chi\chi} + 28J^4{J_\chi}^3J_{\chi\chi\chi} \\
& - 2J^2{J_\chi}^5J_{\chi\chi\chi} + 12J^5J_\chi J_{\chi\chi}J_{\chi\chi\chi} - 3J^3{J_\chi}^3J_{\chi\chi}J_{\chi\chi\chi} + 16J^7J_{\chi\chi\chi\chi} \\
& - 8J^5{J_\chi}^2J_{\chi\chi\chi\chi} + J^3{J_\chi}^4J_{\chi\chi\chi\chi})\chi^6,
\end{aligned} \tag{6.50}$$

which has no ambiguous sign, but is still fiendishly complicated and must be solved numerically, with the exception of the known exact solution (6.37). On this, it is interesting to note that the weakly non-linear ODE (6.50) has a slightly more general form of this exact solution, given by

$$J(\chi) = A\chi^{-2}\mathrm{e}^{B\chi}, \tag{6.51}$$

for *any* constant B. In the fully non-linear case (6.31), we are restricted to $B = \pm 2$.

With regards to numerical solutions, we shall approximate $J(\chi)$ and its derivatives by a spectral Fourier series,

$$J(\chi) = b_0 + \sum_{n=1}^{N} b_n \cos(\alpha_n \chi), \tag{6.52a}$$

$$J_\chi(\chi) = -\sum_{n=1}^{N} b_n \alpha_n \sin(\alpha_n \chi), \tag{6.52b}$$

$$J_{\chi\chi}(\chi) = -\sum_{n=1}^{N} b_n {\alpha_n}^2 \cos(\alpha_n \chi), \tag{6.52c}$$

$$J_{\chi\chi\chi}(\chi) = \sum_{n=1}^{N} b_n {\alpha_n}^3 \sin(\alpha_n \chi), \tag{6.52d}$$

$$J_{\chi\chi\chi\chi}(\chi) = \sum_{n=1}^{N} b_n {\alpha_n}^4 \cos(\alpha_n \chi), \tag{6.52e}$$

where $\alpha_n = n\pi/R$, with R arbitrarily large, and the series are truncated to N terms. Taking the *Galerkin method* approach [46], we define a set of $N+1$ spectral residual functions

$$R_n(b_0, ..., b_N) = \int_0^R G(b_0, ..., b_N) \cos(\alpha_n \chi) \mathrm{d}\chi = 0, \tag{6.53}$$

where $G(b_0, ..., b_N)$ is the J-ODE (6.50), now considered to be a function of the $N+1$ spectral b-coefficients. In defining the set (6.53), we have effectively reduced the problem of finding solutions to the non-linear fourth order ODE (6.50) to the algebraic problem of finding an $N+1$ set of b_n that solve the $N+1$ equations (6.53). The solution of the residuals can be performed by *MATLAB*'s `fsolve` routine, which necessarily requires an initial guess for b_n. Using our intuition as to what we would expect a physically reasonable spherically symmetric charge distribution to look like, we adopt a Gaussian function as our initial guess

$$J_{\text{init}}(\chi) = B\exp\left[-\frac{(\chi-\mu)^2}{2\sigma^2}\right], \tag{6.54}$$

where B, μ and σ (not to be confused with the Dirac scalar σ) are the amplitude, shift and width factors respectively. Incidentally, a useful feature of the Gaussian function is that the three arbitrary parameters can be tweaked until the numerical method starts to converge. The b_0 and b_n coefficients corresponding to this initial guess are [30]

$$b_0 = \frac{B}{R}\int_0^R \exp\left[-\frac{(\chi-\mu)^2}{2\sigma^2}\right] \mathrm{d}\chi, \tag{6.55}$$

$$b_n = \frac{2B}{R}\int_0^R \exp\left[-\frac{(\chi-\mu)^2}{2\sigma^2}\right] \cos(\alpha_n \chi)\mathrm{d}\chi, \tag{6.56}$$

where the integrals are evaluated numerically, using the Legendre-Gaussian quadrature routine `lgwt` by von Winckel.

6.1.7 Numerical solutions of the weakly and fully non-linear ODEs

In our *MATLAB* program for the solution of the weakly non-linear ODE (6.50), we initially set the number of spectral terms $N = 61$, and the number of radial points $P = 1000$, with the maximum radius set to $R = 25$. Setting the initial guess Gaussian function parameters in (6.54) to $B = 2$, $\mu = 0$, and $\sigma = 4.1$, `fsolve` with default function tolerance (`'TolFun'=1e-6`) converged to the solution shown in Figure 6.1.

Using exactly the same parameters, but with the number of spectral terms increased to $N = 101$, we obtain the same solution as in the $N = 61$ case, with negligible difference. Setting $N = 60$ and $N = 100$, keeping everything else the same, the residuals converge to slightly different distributions of similar size. Using the function obtained from the $N = 61$ case as a guide, we make a fresh initial guess with the new Gaussian parameters $B = 0.7$, $\mu = 6$ and $\sigma = 1.5$, with `'TolFun=1e-12'`. This initial guess turned out to be much more accurate, converging to the solution shown in Figure 6.2, but stalling due to the minimum function tolerance being reached. The same routine was run with $N = 101$, which yielded the same solution. For the $N = 101$ case, the residuals reduced to $< \pm 5 \times 10^{-6}$ from values on the order of 10^4 based on the initial Gaussian guess in Figure 6.2. The values of the right hand side of the implicit ODE itself (6.50) were reduced from an order of 10^4 to $< \pm 0.2$, where ideally they should be zero at all radial points. Such a dramatic relative difference in the Galerkin residuals and implicit ODE values between the two plots in Figure 6.2 strongly suggests the presence of a real solution for (6.50), or at least an extreme local minimum in the solution phase space.

With a solution to the weakly non-linear ODE (6.50) in hand, we turn now to finding solutions to the fully non-linear system (6.31), which is the same as the weaker equation, except that it lacks the square root term with ambiguous sign. At this point, we assume that the $J(\chi)$ solution in Figure 6.2 provides a good initial guess to finding solutions of the full ODE. Our initial guess takes the form

$$\bar{\jmath}_a = \epsilon J. \tag{6.57}$$

Setting ϵ to a "small" value of 10^{-2}, we find that the initial guess J immediately satisfies `fsolve` without any further iterations. Therefore for small ϵ, (6.57) appears to be a good approximate solution to (6.31). This result is not unexpected, since the square root term in (6.31) is negligibly small in the limit of small ϵ. Interestingly, it turns out that if we choose $\epsilon = 1$, which is not "small", so that our initial guess is $\bar{\jmath}_{a,\mathrm{init}} = J$, we still obtain a solution for the fully non-linear ODE system, after many iterations. The solutions for both sign cases obtained using the parameters $N = 101$, $P = 1000$, $\epsilon = 1$, and `'TolFun'=1e-12` are displayed in Figure 6.3. Comparing these solutions with the $J(\chi)$ solution in Figure 6.2, we can see that the initial and final guesses for $\bar{\jmath}_a$ closely match, despite the fact that the residual values have been reduced from $\sim 10^4$ to $\sim 10^{-4}$, and the ODE values (this time corresponding to the fully non-linear case) have been reduced from $\sim 10^4$ to ~ 2 through the action of `fsolve`. Now, due to the presence of the square root term in (6.31), a small imaginary part on the order of $\sim 10^{-4}$ to $\sim 10^{-3}$ times the magnitude of the real part is present in the final forms for $\bar{\jmath}_a$, $\bar{\jmath}_{a,\chi}$, etc. We are assuming that all imaginary components are erroneous artefacts produced by the `fsolve` algorithm when dealing with the square root term, and are therefore ignored.

The solutions displayed in Figure 6.3 appear to be static "soliton-like" solutions to the Maxwell-Dirac equations under spherical symmetry. These solutions are of course, not technically solitons since they do not evolve with time, but radially they closely approximate a characteristic $sech^2(\chi)$ soliton profile [47], and have the feature of being localized due to the non-linearity of the system self-interaction. Physically, these solutions correspond to a static, hollow shell of electric charge.

6.1.8 Numerical solutions using multiple Gaussian initial guess

In principle, we can have as many Gaussian functions in our initial guess for $J(\chi)$ as we want, which is useful when searching for solutions with multiple peaks. The corresponding generalizations of (6.54)-(6.56) are

$$J_{\text{init}}(\chi) = \sum_{m=1}^{M} B_m \exp\left[-\frac{(\chi-\mu_m)^2}{2{\sigma_m}^2}\right], \tag{6.58}$$

$$b_0 = \frac{1}{R}\int_0^R \sum_{m=1}^{M} B_m \exp\left[-\frac{(\chi-\mu_m)^2}{2{\sigma_m}^2}\right] \mathrm{d}\chi, \tag{6.59}$$

$$b_n = \frac{2}{R}\int_0^R \sum_{m=1}^{M} B_m \exp\left[-\frac{(\chi-\mu_m)^2}{2{\sigma_m}^2}\right] \cos(\alpha_n \chi)\mathrm{d}\chi. \tag{6.60}$$

Considering the case where $M = 2$, we now have six parameters to choose from when setting J_{init}. After some experimentation, we find that choosing the parameters $B_1 = B_2 = 0.23$, $\mu_1 = 5.0$, $\mu_2 = 8.4$, $\sigma_1 = \sigma_2 = 0.8$, with `'TolFun'` and `'TolX'` both set to `1e-7`, yields the weakly non-linear solution shown in Figure 6.4. The solution curve corresponds to a reduction of the residuals from $\sim 10^2$ to $\sim 10^{-6}$, and a reduction of the magnitude of the ODE values (as a function of χ) from $\sim 10^2$ to $\sim 10^{-2}$, compared with the initial double-Gaussian curve. Bootstrapping this weakly non-linear $J(\chi)$ solution by using it as the initial guess for the fully non-linear scheme, yields $\bar{j}_a(\chi)$ solutions for both positive and negative square root sign cases, with negligible difference to the $J(\chi)$ form in Figure 6.4.

A comparative plot of single and double hump $\bar{j}_a(\chi)$ distributions is given in Figure 6.5. From this, an obvious question arises: does there exist some set of solution eigenvectors $\bar{j}_{a,n}(\chi)$, whereby the number of peaks increases as the order n increases? It is interesting to speculate in this manner, however the analytical extraction of such a set of eigenvectors from (6.31) or (6.50), if any actually exist, would be quite a formidable task indeed. We have demonstrated numerically that solutions of multiple "orders" do exist, but there are serious efficiency problems associated with our simple calculational scheme. Searching for multiple hump solutions with $M > 1$ is quite computationally intensive, with calculations often having to run for several days on end. It may be the case that using a set of basis functions that are more suited to the symmetry of the problem, such as Laguerre or Chebyshev polynomials, will improve efficiency, as well as providing greater insight into the nature of the problem.

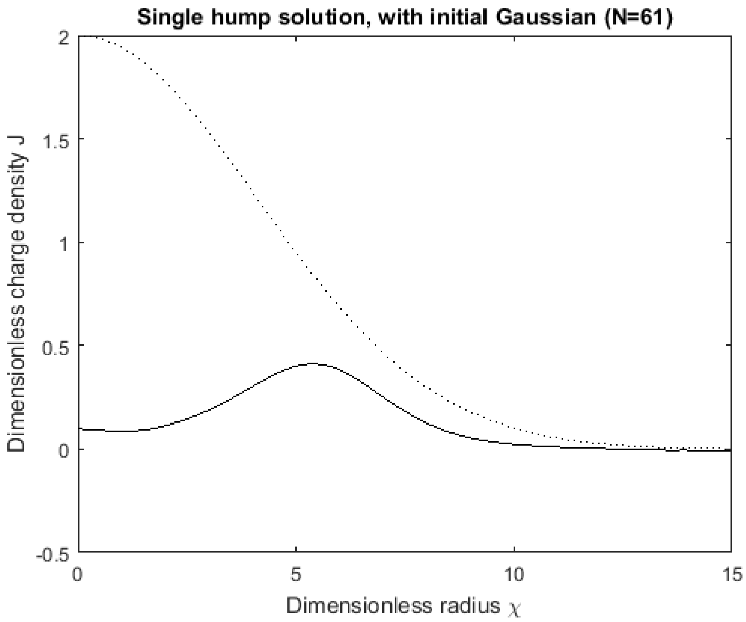

Figure 6.1: $J(\chi)$ solution, given parameters $N = 61$, $B = 2$, $\mu = 0$, $\sigma = 4.1$, and $R = 25$. Default tolerance. Dotted line is the initial Gaussian guess, solid line is the solution function.

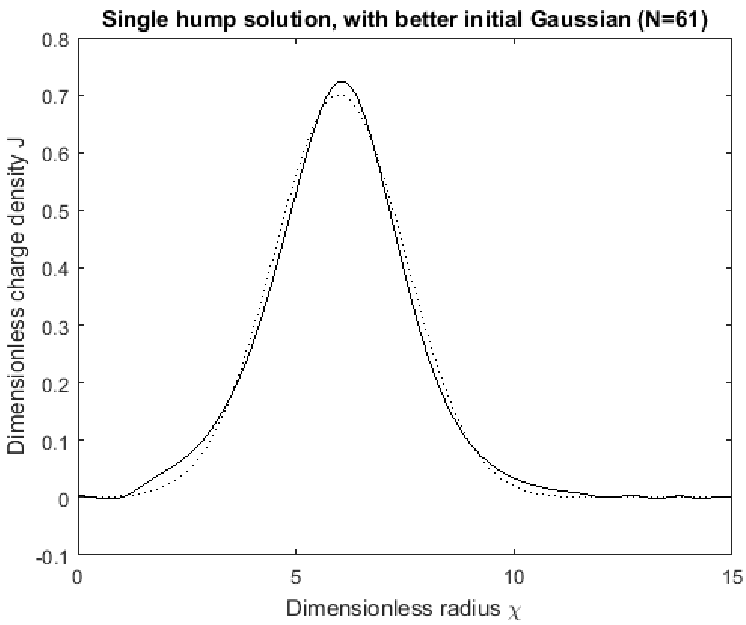

Figure 6.2: $J(\chi)$ solution, given parameters $N = 61$, $B = 0.7$, $\mu = 6$, $\sigma = 1.5$, and $R = 25$. Function tolerance of 10^{-12}. Dotted line is the initial Gaussian guess, solid line is the solution function.

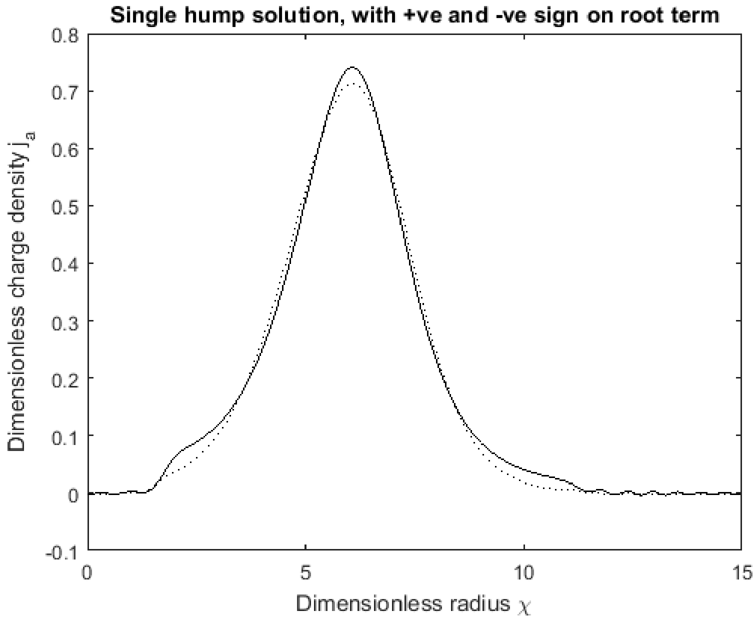

Figure 6.3: Dimensionless $\overline{j}_a(\chi)$ solutions to the static, spherically symmetric Maxwell-Dirac equation ODE, where the solid and dotted lines correspond to positive and negative signs on the square root term respectively.

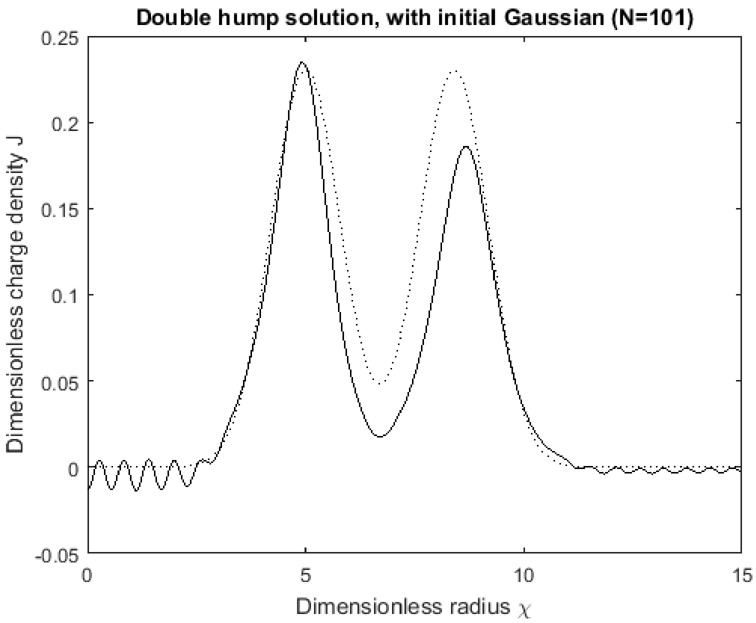

Figure 6.4: $J(\chi)$ solution, given parameters $N = 101$, $B_1 = B_2 = 0.23$, $\mu_1 = 5.0$, $\mu_2 = 8.4$, $\sigma_1 = \sigma_2 = 0.8$ and $R = 25$. Function and step size tolerance of 10^{-7}. Dotted line is the initial $M = 2$ double-Gaussian guess, solid line is the solution function for $J(\chi)$, and a close approximation of the full $\overline{j}_a(\chi)$ solution.

6.2 Physical quantities of the numerical solutions

We now turn our attention to the problem of calculating physical quantities of interest, associated with the numerical solutions obtained in the previous section. In particular, we focus on obtaining explicit quantities for the mass-energy and total charge of the single hump solution given in Figure 6.3. In order to calculate the energy density T_{00}, we must further restrict the spherically symmetric form of the stress-energy tensor given in section 5.3 to include time translation invariance.

6.2.1 Static spherical stress-energy tensor

Setting the time derivatives in the field strength coefficient functions (5.87) and (5.88) to zero, gives

$$\begin{aligned} F_a(r) &= \frac{1}{qr}(\sigma^2-\omega^2)^{-2}\{-2m\sigma j_a(\sigma\sigma_r-\omega\omega_r)\pm \mathrm{i}[\sigma\omega(\sigma_r^2+\omega_r^2)-(\sigma^2+\omega^2)\sigma_r\omega_r]\} \\ &\quad+\frac{1}{qr}(\sigma^2-\omega^2)^{-1}\left[m(\sigma_r j_a+\sigma j_{a,r})\pm\frac{\mathrm{i}}{2}(\sigma\omega_{rr}-\sigma_{rr}\omega)\right], \qquad (6.61) \\ F_b(r) &= \pm\frac{1}{2qr^3}. \qquad (6.62) \end{aligned}$$

Similarly, when taking into account the result $j_b = 0$, the coefficient functions of the stress-energy tensor (5.93)-(5.95) become

$$\begin{aligned} T_a(r) &= (\sigma^2-\omega^2)^{-1}\left[\pm\frac{\mathrm{i}}{2}j_a(\sigma_r\omega-\sigma\omega_r)-m\sigma j_a^2\right], \qquad (6.63) \\ T_b(r) &= 0, \qquad (6.64) \\ T_c(r) &= \pm\frac{\mathrm{i}}{2}j_a(\sigma^2-\omega^2)^{-1}(\sigma_r\omega-\sigma\omega_r)-2\mathcal{F}, \qquad (6.65) \end{aligned}$$

where the form of the Maxwell field energy density (5.92) is the same as before. From (5.89), the energy density in this case is

$$T_{00}(r) = (\sigma^2-\omega^2)^{-1}\left[\pm\frac{\mathrm{i}}{2}j_a(\sigma_r\omega-\sigma\omega_r)-m\sigma j_a^2\right]+\frac{r^2(F_a^2+F_b^2)}{2}. \qquad (6.66)$$

Following our previous work on this symmetry case, in order to make these equations compatible with the dimensionless $\overline{j}_a$ solutions we have obtained, we must rewrite everything in terms of $\overline{j}_a$. As before, we want to convert to real fields, so we apply (6.13), which causes terms quadratic in ω to change sign, and linear terms to absorb free i factors. With this change in convention, (6.61) and (6.63) become

$$\begin{aligned} F_a(r) &= \frac{1}{qr}(\sigma^2+\varpi^2)^{-2}\{-2m\sigma j_a(\sigma\sigma_r+\varpi\varpi_r)\pm[\sigma\varpi(\sigma_r^2-\varpi_r^2)-(\sigma^2-\varpi^2)\sigma_r\varpi_r]\} \\ &\quad+\frac{1}{qr}(\sigma^2+\varpi^2)^{-1}\left[m(\sigma_r j_a+\sigma j_{a,r})\pm\frac{1}{2}(\sigma\varpi_{rr}-\sigma_{rr}\varpi)\right], \qquad (6.67) \\ T_a(r) &= (\sigma^2+\varpi^2)^{-1}\left[\pm\frac{1}{2}j_a(\sigma_r\varpi-\sigma\varpi_r)-m\sigma j_a^2\right]. \qquad (6.68) \end{aligned}$$

Following a careful term-by-term algebraic rearrangement in *Mathematica*, using the expressions given in section 6.1.1 for σ, ϖ and their χ-derivatives in terms of $\overline{j}_a$, our energy density coefficient functions become

$$T_a(r) = \pm\frac{6j_a^2 - 4m^2r^2j_a^2 + 4rj_aj_{a,r} + 2r^2j_{a,r}^2 - r^2j_aj_{a,rr}}{4r\sqrt{m^2r^2j_a^2 - j_a^2 - rj_aj_{a,r} - (1/4)r^2j_{a,r}^2}}, \tag{6.69}$$

$$\begin{aligned} F_a(r) = \pm\,&(-24j_a^5 + 32m^2r^2j_a^5 - 36rj_a^4j_{a,r} + 8m^2r^3j_a^4j_{a,r} - 16r^2j_a^3j_{a,r}^2 + 8m^2r^4j_a^3j_{a,r}^2 \\ &- 18r^3j_a^2j_{a,r}^3 + 12m^2r^5j_a^2j_{a,r}^3 - 12r^4j_aj_{a,r}^4 - 2r^5j_{a,r}^5 - 8m^2r^4j_a^4j_{a,rr} \\ &+ 20r^3j_a^3j_{a,r}j_{a,rr} - 16m^2r^5j_a^3j_{a,r}j_{a,rr} + 14r^4j_a^2j_{a,r}^2j_{a,rr} + 2r^5j_aj_{a,r}^3j_{a,rr} \\ &+ 2r^4j_a^3j_{a,rr}^2 + r^5j_a^2j_{a,r}j_{a,rr}^2 - 4r^3j_a^4j_{a,rrr} + 4m^2r^5j_a^4j_{a,rrr} - 4r^4j_a^3j_{a,r}j_{a,rrr} \\ &- r^5j_a^2j_{a,r}^2j_{a,rrr})\{16qr^3j_a^2[m^2r^2j_a^2 - j_a^2 - rj_aj_{a,r} - (1/4)r^2j_{a,r}^2]^{3/2}\}^{-1}, \end{aligned} \tag{6.70}$$

and (6.62), which is independent of $\overline{j}_a$.

6.2.2 Dimensionless energy density

Introducing the dimensionless objects (6.26)-(6.28), and their obvious extension to higher-order derivatives, we find that

$$T_a = q^{-2}m^4\overline{T}_a, \tag{6.71}$$
$$F_a = q^{-1}m^3\overline{F}_a, \tag{6.72}$$
$$F_b = q^{-1}m^3\overline{F}_b, \tag{6.73}$$

where the dimensionless functions are

$$\overline{T}_a(\chi) = \pm\frac{6\overline{j}_a^2 - 4\chi^2\overline{j}_a^2 + 4\chi\overline{j}_a\overline{j}_{a,\chi} + 2\chi^2\overline{j}_{a,\chi}^2 - \chi^2\overline{j}_a\overline{j}_{a,\chi\chi}}{4\chi\sqrt{\chi^2\overline{j}_a^2 - \overline{j}_a^2 - \chi\overline{j}_a\overline{j}_{a,\chi} - (1/4)\chi^2\overline{j}_{a,\chi}^2}}, \tag{6.74}$$

$$\begin{aligned} \overline{F}_a(\chi) = \pm\,&(-24\overline{j}_a^5 + 32\chi^2\overline{j}_a^5 - 36\chi\overline{j}_a^4\overline{j}_{a,\chi} + 8\chi^3\overline{j}_a^4\overline{j}_{a,\chi} - 16\chi^2\overline{j}_a^3\overline{j}_{a,\chi}^2 + 8\chi^4\overline{j}_a^3\overline{j}_{a,\chi}^2 \\ &- 18\chi^3\overline{j}_a^2\overline{j}_{a,\chi}^3 + 12\chi^5\overline{j}_a^2\overline{j}_{a,\chi}^3 - 12\chi^4\overline{j}_a\overline{j}_{a,\chi}^4 - 2\chi^5\overline{j}_{a,\chi}^5 - 8\chi^4\overline{j}_a^4\overline{j}_{a,\chi\chi} \\ &+ 20\chi^3\overline{j}_a^3\overline{j}_{a,\chi}\overline{j}_{a,\chi\chi} - 16\chi^5\overline{j}_a^3\overline{j}_{a,\chi}\overline{j}_{a,\chi\chi} + 14\chi^4\overline{j}_a^2\overline{j}_{a,\chi}^2\overline{j}_{a,\chi\chi} + 2\chi^5\overline{j}_a\overline{j}_{a,\chi}^3\overline{j}_{a,\chi\chi} \\ &+ 2\chi^4\overline{j}_a^3\overline{j}_{a,\chi\chi}^2 + \chi^5\overline{j}_a^2\overline{j}_{a,\chi}\overline{j}_{a,\chi\chi}^2 - 4\chi^3\overline{j}_a^4\overline{j}_{a,\chi\chi\chi} + 4\chi^5\overline{j}_a^4\overline{j}_{a,\chi\chi\chi} - 4\chi^4\overline{j}_a^3\overline{j}_{a,\chi}\overline{j}_{a,\chi\chi\chi} \\ &- \chi^5\overline{j}_a^2\overline{j}_{a,\chi}^2\overline{j}_{a,\chi\chi\chi})\{16\chi^3\overline{j}_a^2[\chi^2\overline{j}_a^2 - \overline{j}_a^2 - \chi\overline{j}_a\overline{j}_{a,\chi} - (1/4)\chi^2\overline{j}_{a,\chi}^2]^{3/2}\}^{-1}, \end{aligned} \tag{6.75}$$

$$\overline{F}_b(\chi) = \pm\frac{1}{2\chi^3}. \tag{6.76}$$

In terms of these dimensionless functions, the dimensionless energy density is

$$\overline{T}_{00} \equiv q^2m^{-4}T_{00} = \overline{T}_a + (\chi^2/2)(\overline{F}_a^2 + \overline{F}_b^2). \tag{6.77}$$

6.2.3 Total mass and charge in static spherical case

From Weinberg [45], we know that the total four-momentum is obtained via the spatial integral

$$p^{\nu} = \int T^{0\nu} \mathrm{d}^3 x, \tag{6.78}$$

where the total energy is

$$p^0 \equiv E = \sqrt{M^2 + |\boldsymbol{p}|^2}. \tag{6.79}$$

Substituting (6.64) into (5.90), we find that

$$T^{0i} = 0, \tag{6.80}$$

so the only non-zero component of the four-momentum is

$$p^0 = M = \int T^{00} \mathrm{d}^3 x. \tag{6.81}$$

Since the integrand is a function of radius only, we should rewrite the volume integral in terms of spherical coordinates

$$M = \iint r^2 T^{00}\, \mathrm{d}r \mathrm{d}\Omega = 4\pi \int r^2 T^{00}\, \mathrm{d}r. \tag{6.82}$$

Incidentally, the stress-energy tensor must also satisfy the conservation constraint

$$\partial_{\mu} T^{\mu\nu} = 0, \tag{6.83}$$

the $\nu = 0$ component of which is

$$\partial_0 T^{00} = \partial_t T^{00} = 0. \tag{6.84}$$

We can see this is automatically satisfied by our symmetry constraint. We are not interested in the pure spatial components of T^{ij} at the moment. Now, rewriting (6.82) in terms of dimensionless objects, we find the obvious result

$$M = m\overline{M}, \tag{6.85}$$

where the dimensionless mass parameter is

$$\overline{M} = 4\pi q^{-2} \int \chi^2 \overline{T}_{00}\, \mathrm{d}\chi = \alpha^{-1} \int \chi^2 \overline{T}_{00}\, \mathrm{d}\chi, \tag{6.86}$$

and we have used the fact that

$$T^{00} = \eta^{0\mu} \eta^{0\nu} T_{\mu\nu} = T_{00}. \tag{6.87}$$

Similarly, the total charge is given by

$$Q = \int j^0\, \mathrm{d}^3 x = 4\pi \int r^2 j_a\, \mathrm{d}r. \tag{6.88}$$

Converting to dimensionless variables, we find that

$$Q = \overline{Q} = \alpha^{-1} \int \chi^2 \overline{j}_a \, \mathrm{d}\chi, \tag{6.89}$$

where the equivalence of Q and $\overline{Q}$ is expected, since charge is dimensionless in natural L-H units. For future studies on static spherical solutions to the Maxwell-Dirac equations, we could constrain the total charge to be an integer multiple of the elementary charge,

$$Q = ne. \tag{6.90}$$

The constraint condition in this case would be

$$\int \chi^2 \overline{j}_a \, \mathrm{d}\chi = ne\alpha. \tag{6.91}$$

6.2.4 Mass and charge of numerical solutions

We now focus on the numerical calculation of the mass and charge corresponding to the two single hump solutions in 6.3. Since the two solutions are so similar in shape and size, we would expect their respective mass-energy and charge integrals to also be similar, so we take these cases to be approximately equal from the outset. This approximation is especially accurate for smaller $\overline{j}_a$ distributions, where the square root term in the fully non-linear Maxwell-Dirac ODE (6.31) is negligible, and the weakly non-linear part (6.50) is dominant.

There is a difficulty in the numerical evaluation of (6.86) for our $\overline{j}_a$ distribution however, in that the integrand behaves badly at the edges of the hump, and in the $\overline{j}_a \approx 0$ region. The source of this bad behavior is apparent when the forms of $\overline{T}_a$ and $\overline{F}_a$ in (6.74) and (6.75) respectively are observed to have

$$\chi\overline{\sigma} = \sqrt{\chi^2 \overline{j}_a^2 - \overline{j}_a^2 - \chi \overline{j}_a \overline{j}_{a,\chi} - (1/4)\chi^2 \overline{j}_{a,\chi}^2} \tag{6.92}$$

terms in the denominators, where $\overline{\sigma}$ is the dimensionless analogue of (6.5). Obviously, the $\overline{M}$ integrand will be "badly behaved" in regions where the square root argument in $\chi\overline{\sigma}$ is ≤ 0. A plot of the real part (again, assuming the small imaginary part is a numerical artefact) of (6.92) for our single-hump $\overline{j}_a$ solution is given in Figure 6.6, along with a vertically exaggerated plot of $\overline{j}_a$ itself. The vertical lines indicate where the real part of $\chi\overline{\sigma} \approx 0$, and therefore correspond to a natural choice of integration interval. Notice how in Figure 6.6, a small portion of the $\overline{j}$ interval lies in the region where $\mathrm{Re}(\chi\overline{\sigma}) = 0$, and is excluded from the mass integral. Numerically, this should not change the mass integral very much, due to the smallness of $\overline{j}_a$ and χ. What this minor overlap of the $\overline{j}_a$ distribution with an area where $\chi\overline{\sigma}$ is ill-defined tells us about the validity of the $\overline{j}_a$ solution itself is unclear at this stage. Formally, we should require that

$$\chi^2 \overline{j}_a^2 - \overline{j}_a^2 - \chi \overline{j}_a \overline{j}_{a,\chi} - (1/4)\chi^2 \overline{j}_{a,\chi}^2 \geq 0, \tag{6.93}$$

however this level of analytical detail is beyond the scope of this preliminary numerical study, but should be addressed in the future.

Now, when we actually perform the mass-energy integral (6.86), we encounter another problem. We find that near the edges of the $\chi\overline{\sigma}$ region, the magnitude of the integrand $\chi^2\overline{T}_{00}$ becomes very large, as can be seen in Figure 6.7. Observing the behavior of $\overline{j}_{a,\chi\chi\chi}$ near the edges of the integrand, we can see some numerical instability present, which becomes more severe closer to the edges. The lower-order derivatives display a similar, but less extreme, kind of instability. The source of the spikes in the integrand becomes apparent when we look at the detailed form of (6.77), particularly the relative orders of $\overline{j}_a$ and its derivatives in the numerators and denominators of (6.74) and (6.75). Looking at the $\overline{T}_a$ expression (6.74) first, we can see that the denominator terms are $O(\overline{j}_a)$ and the numerator terms are $O(\overline{j}_a^2)$. So roughly speaking, we would expect that as $\chi\overline{\sigma} \to 0$, the numerator of $\overline{T}_a$ should approach zero more rapidly, so that $\overline{T}_a \to 0$. This is provided of course, that the derivative terms in the numerator of (6.74) smoothly approach zero as $\chi\overline{\sigma} \to 0$, which they do not due to the aforementioned numerical instability, the result being the large spikes in Figure 6.7.

Turning now to the Maxwell terms $\overline{F}_a$ and $\overline{F}_b$ in (6.75) and (6.76) respectively, we can see that they both have the same order of $\overline{j}_a$ in the numerator and denominator. This is obviously the case in the magnetic $\overline{F}_b$ term, which is a function of χ only, but it should also be the case in the electric term, in regions where $\overline{F}_a \sim \text{const.}/\chi^3$. As in the $\overline{T}_a$ case, in order for $\overline{F}_a$ to be well behaved as $\chi\overline{\sigma} \to 0$, it should approach this $\propto \chi^{-3}$ form, with the numerator vanishing as rapidly as $\chi\overline{\sigma}$. But again, the numerical instability near the edges of the mass integrand destroys this behavior.

Note that this $f(\chi)$ form of the Maxwell terms implies they are technically non-zero in regions where $\chi\overline{\sigma} = 0$. This is obviously the case with the monopole term $\overline{F}_b$, whose contribution to the mass integrand is proportional to χ^{-2}, meaning there is a singularity in $\overline{T}_{00}$ at the origin. From this point of view, it appears that strictly restricting the integration region to where $\chi\overline{\sigma} \geq 0$ is a physically sensible choice.

It turns out that we can eliminate the peaks in the mass integrand entirely, as shown in Figure 6.8, if we manually set the derivatives to zero at arbitrary points near the edges of the distribution. The cut-off points are chosen such that they lie before the worst of the instability, but where the derivative values are small, and would presumably vanish anyway in the absence of instability.

Performing the numerical mass integration (6.86), we find that the dimensionless mass with the large peaks intact, as in Figure 6.7, is $\overline{M} \approx 1.3 \times 10^8$, whereas the "revised" mass integration (Figure 6.8) gives $\overline{M} \approx 1.4 \times 10^5$. There is a residual imaginary part for each of these values, on the order of 10^3 and 10^{-3} for each result respectively. The conversion to the dimensional form is given by multiplying by the electron mass in natural units, which is $m_e \approx 0.511$ MeV, so that our two masses become $M \approx 6.7 \times 10^7$ MeV and $M \approx 7.3 \times 10^4$ MeV respectively.

These values are extremely large, on the order of tens of TeV and GeV for each case, so that their physical relevance is dubious. Therefore, this solution should be treated as primarily of theoretical interest. Further searches for solutions should focus on finding much smaller $\overline{j}_a$ distributions, closer to the origin. Incidentally, this is where the so-called "weakly non-linear" scheme dominates, and the contribution of the difficult square root term in (6.31) becomes negligible. These "physically relevant" solutions have been difficult to locate convincingly so far, but there are further constraints we can impose that could improve prospects. The condition that $\overline{\sigma}$ is real (6.93), as well as the charge quantization constraint (6.91) should both be included as global numerical constraints in the algorithm for the solution of (6.31).

Finally, we perform the numerical charge integration (6.89), which is free of the issues affecting the mass integral calculation. The value obtained is $Q \approx 1.3 \times 10^4$ in dimensionless natural units. The conversion from charge in natural L-H units to SI units is determined by the ratio

$$\frac{e}{\sqrt{4\pi\alpha}} = 1 = 5.29 \times 10^{-19}\ \mathrm{C}, \tag{6.94}$$

which tells us that 1 unit of dimensionless charge corresponds to 5.29×10^{-19} of charge in Coulombs. So the charge of our $\overline{j}_a$ solution is

$$Q \approx 7.1 \times 10^{-15}\ \mathrm{C} \approx 4.4 \times 10^4\, e, \tag{6.95}$$

which like the mass, is extremely large.

6.3 Solutions in the $\widetilde{P}_{13,10}$ "trans-boost" case

As we saw in section 4.4, the Maxwell-Dirac system under $\widetilde{P}_{13,10}$ subgroup symmetry (or alternatively, "trans-boost" symmetry, due to the simultaneous translation/boost nature of the $\widetilde{B}_\lambda$ operator), consists of the algebraic equation

$$\lambda^6 q^4 (j_a j_b)^2 \mp 4\lambda^4 q^2 m (j_a j_b)^{3/2} + 4(\lambda^2 m^2 - 1) j_a j_b - 4k_a k_b + k_d^2 = 0, \tag{6.96}$$

along with the constraint

$$j_a k_b = -j_b k_a. \tag{6.97}$$

λ is a free parameter > 0, intrinsic to the subgroup. Hence the trans-boost "subgroup" is really an infinite family of subgroups, each one with a particular λ value. We also have the globally constant parameters

$$\sigma = \pm 2\sqrt{j_a j_b}, \tag{6.98}$$

$$k_c = \lambda^3 q^2 \sigma^2/4 - \lambda m \sigma = \lambda^3 q^2 j_a j_b \mp 2\lambda m \sqrt{j_a j_b}, \tag{6.99}$$

and k_d, whose particular value is not determined by the other parameters. As we shall see later on, the *range* of allowed values of k_d *does* depend on the other parameters. It is important to note that all of our parameters are real, global constants, and that our approach to finding solutions will consist of defining the appropriate limits and interrelationships of our parameter set, thereby defining algebraic solution domains. Due to the algebraic nature of the Maxwell-Dirac equations under trans-boost subgroup symmetry, our analysis in this case will be quite different than the static spherically analogue presented in the previous section. The most striking difference is the relative ease with which we can obtain analytical closed form solutions, whereas solutions to (6.31) almost certainly have to be obtained numerically.

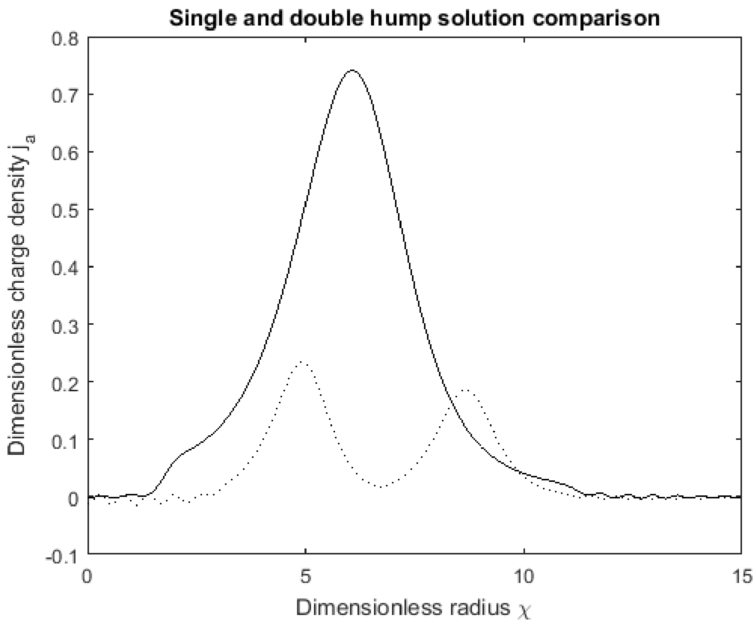

Figure 6.5: A comparison of $\overline{j}_a(\chi)$ obtained from single and double hump Gaussian initial guesses.

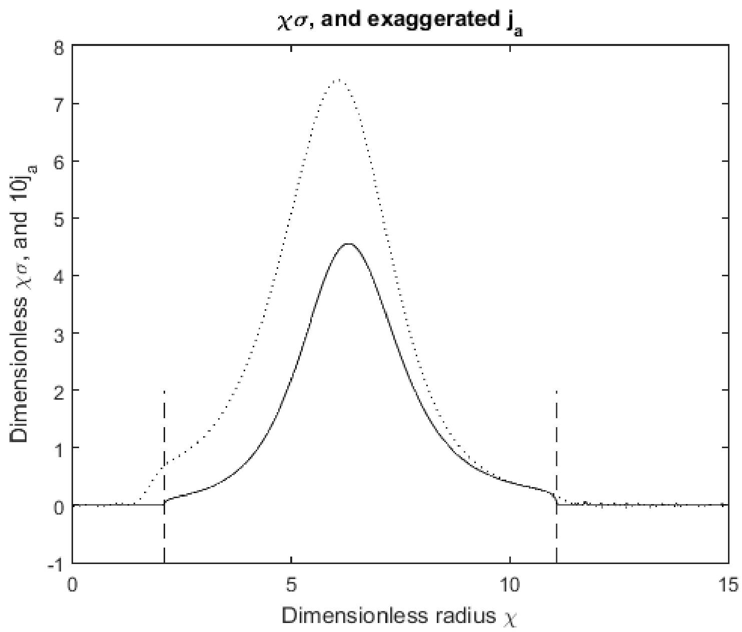

Figure 6.6: Comparison of $\chi\overline{\sigma}$ (solid line) with a $10\times$ vertically exaggerated $\overline{j}_a$ plot (dotted line). The outer boundary of the integration region is indicated by the vertical dashed lines, and is chosen to be where the real part of $\chi\overline{\sigma}$ is non zero.

For convenience, we will also repeat the trans-boost invariant forms of the current four-vectors, as well as the non-zero components of the field strength tensor. The two four-vectors are

$$j^\mu = \begin{pmatrix} j_a \mathrm{e}^{-2y/\lambda} + j_b \mathrm{e}^{2y/\lambda} \\ 0 \\ 0 \\ j_a \mathrm{e}^{-2y/\lambda} - j_b \mathrm{e}^{2y/\lambda} \end{pmatrix}, \quad k^\mu = \begin{pmatrix} k_a \mathrm{e}^{-2y/\lambda} + k_b \mathrm{e}^{2y/\lambda} \\ k_c \\ k_d \\ k_a \mathrm{e}^{-2y/\lambda} - k_b \mathrm{e}^{2y/\lambda} \end{pmatrix}, \tag{6.100}$$

and the two non-zero field strength components are

$$F_{02} = E_y = F_a \mathrm{e}^{-2y/\lambda} - F_b \mathrm{e}^{2y/\lambda}, \tag{6.101}$$
$$F_{23} = -M_x = F_a \mathrm{e}^{-2y/\lambda} + F_b \mathrm{e}^{2y/\lambda}, \tag{6.102}$$

where the globally constant components are

$$F_a = -\frac{2j_a(m\lambda\sigma + k_c)}{q\lambda^2\sigma^2}, \tag{6.103}$$
$$F_b = -\frac{2j_b(m\lambda\sigma + k_c)}{q\lambda^2\sigma^2}. \tag{6.104}$$

E_y and M_x are the y and x components of the electric and magnetic fields respectively.

6.3.1 Non-dimensionalized equations

The non-dimensionalization procedure is directly analogous to that followed in section 6.1.2, as we are still working in natural Lorentz-Heaviside (L-H) units. Remembering that in these units, charge is dimensionless, and noting that the dimensions of the subgroup parameter must be the same as those for the y coordinate

$$[\lambda] = [y] = [L], \tag{6.105}$$

the dimensionality of the parameters that appear in our physical equations are

$$[j_a] = [j_b] = [k_a] = [k_b] = [k_d] = [\sigma] = [L]^{-3}. \tag{6.106}$$

We therefore define the dimensionless parameters

$$\overline{\lambda} = m\lambda, \tag{6.107}$$
$$\overline{y} = my, \tag{6.108}$$
$$\overline{j}_a = q^2 m^{-3} j_a, \tag{6.109}$$

and so on for the other five parameters. Replacing the dimensional objects in (6.96)-(6.99) with their dimensionless counterparts, and discarding any common m and q dependent factors, gives the dimensionless trans-boost invariant Maxwell-Dirac system

$$\overline{\lambda}^6(\overline{j}_a\overline{j}_b)^2 \mp 4\overline{\lambda}^4(\overline{j}_a\overline{j}_b)^{3/2} + 4(\overline{\lambda}^2 - 1)\overline{j}_a\overline{j}_b - 4\overline{k}_a\overline{k}_b + \overline{k}_d^2 = 0, \tag{6.110}$$

$$\overline{j}_a\overline{k}_b = -\overline{j}_b\overline{k}_a, \tag{6.111}$$
$$\overline{\sigma} = \pm 2\sqrt{\overline{j}_a\overline{j}_b}, \tag{6.112}$$
$$\overline{k}_c = \overline{\lambda}^3\overline{\sigma}^2/4 - \overline{\lambda}\overline{\sigma} = \overline{\lambda}^3\overline{j}_a\overline{j}_b \mp 2\overline{\lambda}\sqrt{\overline{j}_a\overline{j}_b}. \tag{6.113}$$

The dimensional four-vectors (6.100) and the field strength tensor components (6.101) and (6.102) have exactly the same form when non-dimensionalized, and the field strength coefficient functions now become

$$\overline{F}_a = -\frac{2\overline{j}_a(\overline{\lambda}\overline{\sigma} + \overline{k}_c)}{\overline{\lambda}^2\overline{\sigma}^2}, \tag{6.114}$$
$$\overline{F}_b = -\frac{2\overline{j}_b(\overline{\lambda}\overline{\sigma} + \overline{k}_c)}{\overline{\lambda}^2\overline{\sigma}^2}. \tag{6.115}$$

So we find that the dimensionality of $\overline{F}_a$ and $\overline{F}_b$, and hence $\overline{E}_y$ and $\overline{M}_x$ is $[L]^{-2}$, and the dimensionless electromagnetic parameters are defined as

$$\overline{F}_a = qm^{-2}F_a, \tag{6.116}$$

and similarly for $\overline{F}_b$. From here on, we look for solutions by choosing special simplifying cases of $\overline{\lambda}$ and $\overline{k}_d$, then gradually working towards the general case.

6.3.2 Case $\overline{\lambda} = 1$, $\overline{k}_d = 0$

For convenience, let us define the new constant parameters $c_j = \overline{j}_a\overline{j}_b$ and $c_k = \overline{k}_a\overline{k}_b$. These two constants are related via the Maxwell-Dirac equation (6.110), with $\overline{\lambda} = 1$, $\overline{k}_d = 0$:

$$c_k = c_j^2/4 \mp c_j^{3/2}. \tag{6.117}$$

We can replace terms with subscript b by using the identities

$$\overline{j}_b = c_j/\overline{j}_a, \tag{6.118}$$
$$\overline{k}_b = c_k/\overline{k}_a. \tag{6.119}$$

The constraint equation (6.111) then becomes

$$c_j/c_k = -(\overline{j}_a/\overline{k}_a)^2, \tag{6.120}$$

which we shall refer to as the “ratio equation”, which imposes the requirement that c_j and c_k have opposite signs, since $\overline{j}_a$ and $\overline{k}_a$ are real. We require c_k, $\overline{\sigma}$ and $\overline{k}_c$, given by (6.117), (6.112) and (6.113) respectively, to be real. For this to be the case, due to the presence of $\sqrt{c_j}$ terms in all of these expressions, we must also require $c_j > 0$. Since c_j is required to be positive, and of opposite sign to c_k by (6.120), then it must be that $c_k < 0$.

In order for c_k to have any negative range at all, we must choose the *negative* sign in the Maxwell-Dirac equation (6.117) (which corresponds to choosing $\sigma > 0$), so that

$$c_k = c_j^2/4 - c_j^{3/2}. \tag{6.121}$$

A plot of this function is given in Figure 6.9, from which we can see that $c_k < 0$ in the range $0 < c_j < 16$. We can easily prove this analytically, by setting $c_k = 0$ in (6.121), then solving for c_j

$$c_j^{3/2}(c_j^{1/2}/4 - 1) = 0, \tag{6.122}$$

from which we can see that the two solutions are $c_j(c_k = 0) = \{0, 16\}$, verifying that the valid solution domain is

$$0 < c_j < 16. \tag{6.123}$$

We now want to rewrite all of our four-vector parameters in terms of $\overline{j}_a$ and c_j. Replacing c_k in the ratio equation (6.120) by substituting (6.121), then solving for $\overline{k}_a$ gives

$$\overline{k}_a = \pm\overline{j}_a\sqrt{-(c_j/4 - c_j^{1/2})}, \tag{6.124}$$

which is real only when $c_k < 0$. Substituting this and (6.121) into (6.119), we get

$$\overline{k}_b = \mp(c_j/\overline{j}_a)\sqrt{-(c_j/4 - c_j^{1/2})}. \tag{6.125}$$

The other parameters dependent upon c_j are

$$\overline{\sigma} = 2\sqrt{c_j}, \tag{6.126}$$

$$\overline{k}_c = \overline{\sigma}^2/4 - \overline{\sigma} = c_j - 2\sqrt{c_j}, \tag{6.127}$$

and the forms of the dimensionless four-vector currents are

$$\overline{j}^\mu = \begin{pmatrix} \overline{j}_a e^{-2\overline{y}} + (c_j/\overline{j}_a)e^{2\overline{y}} \\ 0 \\ 0 \\ \overline{j}_a e^{-2\overline{y}} - (c_j/\overline{j}_a)e^{2\overline{y}} \end{pmatrix}, \tag{6.128}$$

$$\overline{k}^\mu = \begin{pmatrix} \pm\overline{j}_a\sqrt{-(c_j/4 - c_j^{1/2})}e^{-2\overline{y}} \mp (c_j/\overline{j}_a)\sqrt{-(c_j/4 - c_j^{1/2})}e^{2\overline{y}} \\ c_j - 2\sqrt{c_j} \\ 0 \\ \pm\overline{j}_a\sqrt{-(c_j/4 - c_j^{1/2})}e^{-2\overline{y}} \pm (c_j/\overline{j}_a)\sqrt{-(c_j/4 - c_j^{1/2})}e^{2\overline{y}} \end{pmatrix}. \tag{6.129}$$

The field strength tensor constants are

$$\overline{F}_a = -\frac{\overline{j}_a}{2}, \tag{6.130}$$

$$\overline{F}_b = -\frac{c_j}{2\overline{j}_a}, \tag{6.131}$$

giving the form of the only non-zero electric and magnetic field components

$$\overline{E}_y = -(\overline{j}_a/2)e^{-2\overline{y}} + (c_j/2\overline{j}_a)e^{2\overline{y}}, \tag{6.132}$$

$$\overline{M}_x = (\overline{j}_a/2)e^{-2\overline{y}} + (c_j/2\overline{j}_a)e^{2\overline{y}}. \tag{6.133}$$

A plot of $\overline{j}^0$ and $\overline{j}^3$ vs. $\overline{y}$, for the values $c_j = 9$, $\overline{j}_a = 3$ is given in Figure 6.10, and the corresponding plot of $\overline{E}_y$ and $\overline{M}_x$ vs. $\overline{y}$ is given in Figure 6.11. In this special case, the fields have the forms

$$\overline{j}^0 = 6\cosh(2\overline{y}), \qquad \overline{j}^3 = -6\sinh(2\overline{y}), \tag{6.134}$$

$$\overline{E}_y = 3\sinh(2\overline{y}), \qquad \overline{M}_x = 3\cosh(2\overline{y}). \tag{6.135}$$

The $c_j = 9$ choice corresponds to the minimum c_k value in Figure 6.9, and choosing $\overline{j}_a = 3$ ensures that $\overline{j}_b = 3$ also. Choices of $\overline{j}_a$ less than or greater than 3, shift the $\overline{y}$-intercept of $\overline{j}^3$ left or right respectively, breaking the symmetry about the vertical axis.

From Figure 6.10, we can see that $\overline{j}^0$, the "charge density" is always positive, and grows to infinity at large magnitudes of $\overline{y}$. We can also see that $\overline{j}^3$, the z-component of the "current flux density", vanishes at the origin, and grows rapidly to large positive values for $\overline{y} < 0$, and to large negative values for $\overline{y} > 0$. Conceptually, we can imagine "sheets of charge" in the $x - z$ plane undergoing laminar flow in the z-direction, with flow magnitude being a function of $\overline{y}$ as in (6.134).

Figure 6.11 shows a similar result, with a positive-definite magnetic field x-component growing to large values for large $\overline{y}$ values. The y-component of the electric field is zero at the origin, growing rapidly to large positive values for $\overline{y} > 0$, and large negative values for $\overline{y} < 0$. The hyperbolic distributions (6.134) and (6.135) were discussed independently by Legg [31].

6.3.3 Case $\overline{\lambda} = 1$, general $\overline{k}_d$

Retaining our definitions of c_j and c_k, our Maxwell-Dirac equation with general $\overline{k}_d$ values is

$$c_k = c_j^2/4 \mp c_j^{3/2} + \overline{k}_d^2/4, \tag{6.136}$$

where we treat $\overline{k}_d$ as a free parameter (as we do with $\overline{\lambda}$), since it is not *directly* dependent on either c_j ar c_k. The ratio equation (6.120) still holds, so we require $c_j > 0$ and $c_k < 0$ as before. The terms c_j^2, $c_j^{3/2}$ and $\overline{k}_d^2$ are all positive definite, so the only way that c_k is going to have a negative region is if we again choose the positive sign in (6.136).

In the previous example, we found that when $\overline{k}_d = 0$, $c_k < 0$ when $0 < c_j < 16$. Since $\overline{k}_d^2/4$ is a positive number, we would expect that the allowed range for c_j would in general be smaller than this, and for $\overline{k}_d$ large enough, there would be no allowed range at all. We can look at this valid solution region explicitly, by plotting the $c_k = 0$ contour as a function of c_j and $\overline{k}_d$. This contour is shown in Figure 6.12, and the valid solution domain lies underneath the curve, in the region where $c_k < 0$. We will take the point of view that the allowed values of $\overline{k}_d$ depend on the given c_j values, although the opposite point of view is valid also. Therefore, as a function of c_j (with full range $0 < c_j < 16$), the corresponding maximum allowed range of $\overline{k}_d$ is given by

$$|\overline{k}_d| < 2\sqrt{c_j^{3/2} - (c_j^2/4)}, \tag{6.137}$$

which is obtained by setting $c_k = 0$ in (6.136) and solving as an inequality for $\overline{k}_d$. The maximum allowed range for $\overline{k}_d$ corresponds to the point where the derivative of (6.137) vanishes, which turns out to be $c_j = 9$. Substituting this value into (6.137), we find that $|\overline{k}_d| < 3\sqrt{3}$ at its largest extent.

Now we want to rewrite all of the four-vector parameters in terms of $\overline{j}_a$, c_j and $\overline{k}_d$. Substituting (6.136) (with negative sign only) into (6.120) and solving for $\overline{k}_a$ gives

$$\overline{k}_a = \pm \overline{j}_a \sqrt{-[c_j/4 - c_j^{1/2} + \overline{k}_d^2/4c_j]}, \tag{6.138}$$

and since $\overline{k}_b = c_k/\overline{k}_a$, we also have

$$\overline{k}_b = \mp(c_j/\overline{j}_a)\sqrt{-[c_j/4 - c_j^{1/2} + \overline{k}_d^2/4c_j]}. \tag{6.139}$$

The other constant parameters, $\overline{\sigma}$ and $\overline{k}_c$ are as they are in (6.112) and (6.113). The form of the four-vectors in the case where $\overline{\lambda} = 1$ is

$$\overline{j}^\mu = \begin{pmatrix} \overline{j}_a e^{-2\overline{y}} + (c_j/\overline{j}_a)e^{2\overline{y}} \\ 0 \\ 0 \\ \overline{j}_a e^{-2\overline{y}} - (c_j/\overline{j}_a)e^{2\overline{y}} \end{pmatrix}, \tag{6.140}$$

$$\overline{k}^\mu = \begin{pmatrix} \pm\sqrt{-[c_j/4 - c_j^{1/2} + \overline{k}_d^2/4c_j]}[\overline{j}_a e^{-2\overline{y}} - (c_j/\overline{j}_a)e^{2\overline{y}}] \\ c_j - 2\sqrt{c_j} \\ \overline{k}_d \\ \pm\sqrt{-[c_j/4 - c_j^{1/2} + \overline{k}_d^2/4c_j]}[\overline{j}_a e^{-2\overline{y}} + (c_j/\overline{j}_a)e^{2\overline{y}}] \end{pmatrix}, \tag{6.141}$$

where $\overline{j}^\mu$ has not changed from the $\overline{k}_d = 0$ case. The constant components of the field strength tensor are as they are in (6.130) and (6.131), so the electric and magnetic fields (6.132) and (6.133) are also the same as when $\overline{k}_d = 0$. The reason for this is that these quantities are independent of the value of $\overline{k}_d$.

6.3.4 Case $\overline{k}_d = 0$, general $\overline{\lambda}$

We now switch to the case where $\overline{k}_d = 0$ again, but we allow $\overline{\lambda}$ to assume arbitrary values > 0. Solving for c_k, the Maxwell-Dirac equation is now

$$c_k = \overline{\lambda}^6 c_j^2/4 \mp \overline{\lambda}^4 c_j^{3/2} + (\overline{\lambda}^2 - 1)c_j. \tag{6.142}$$

As usual, the ratio equation (6.120) requires that $c_j > 0$ and $c_k < 0$. Since the $(\overline{\lambda}^2 - 1)c_j$ term is not positive definite, we need to consider the c_k surface for both signs in (6.142), where the negative and positive signs correspond to $\overline{\sigma} > 0$ and $\overline{\sigma} < 0$ respectively. The contour plots for $c_k = 0$ for the two sign cases are shown in Figure 6.13. As before, the $c_k = 0$ contours correspond to the limits of the valid domain for c_j and $\overline{\lambda}$ values, but this time the situation is slightly more complicated, due to the two different sign cases in (6.142). Setting $c_k = 0$ in (6.142), we get

$$c_j[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1)] = 0, \tag{6.143}$$

which has a trivial solution $c_j = 0$. The expression in the brackets is quadratic in $\sqrt{c_j}$, and has the two solutions

$$c_j = 4(\overline{\lambda}^2 \pm 2\overline{\lambda} + 1)/\overline{\lambda}^6, \tag{6.144}$$

which correspond to the $c_k = 0$ contours in Figure 6.13. Consider the case where $\overline{\sigma} < 0$, which corresponds to the positive sign choice in (6.142). Intuitively, we can see that the only way to obtain $c_k < 0$ values is if $\overline{\lambda} < 1$, so the allowed c_j and $\overline{\lambda}$ values for this case lie to the bottom-left of the dashed curve in Figure 6.13. Now, for the negative sign choice ($\overline{\sigma} > 0$), we know from the previous cases that when $\overline{\lambda} = 1$, the allowed range for c_j is $0 < c_j < 16$. Therefore the allowed c_j and $\overline{\lambda}$ values should be bounded at the top by the solid curve in Figure 6.13. From our $c_k = 0$ contour solutions, we find that the $\{c_j, \overline{\lambda}\}$ domain is bounded at the bottom by $c_j = 0$ for $\overline{\lambda} < 1$, and by the dashed $c_k = 0$ contour when $\overline{\lambda} > 1$. For $\overline{\lambda} < 1$, the dashed contour is not applicable to the $\overline{\sigma} > 0$ case.

The allowed solution domains for the various cases can be succinctly summarized as follows. For the $\overline{\sigma} > 0$ case, $c_k < 0$ when

$$\begin{aligned} &0 < \overline{\lambda} < 1, && 0 < c_j < 4(\overline{\lambda}^2 + 2\overline{\lambda} + 1)/\overline{\lambda}^6, \\ &\overline{\lambda} = 1, && 0 < c_j < 16, \\ &\overline{\lambda} > 1, && 4(\overline{\lambda}^2 - 2\overline{\lambda} + 1)/\overline{\lambda}^6 < c_j < 4(\overline{\lambda}^2 + 2\overline{\lambda} + 1)/\overline{\lambda}^6. \end{aligned} \tag{6.145}$$

For the $\overline{\sigma} < 0$ case, $c_k < 0$ when

$$\begin{aligned} &0 < \overline{\lambda} < 1, && 0 < c_j < 4(\overline{\lambda}^2 - 2\overline{\lambda} + 1)/\overline{\lambda}^6, \\ &\overline{\lambda} \geq 1, && \text{No real solutions.} \end{aligned} \tag{6.146}$$

As usual, we want to rewrite all of our four-vector parameters in terms of $\overline{j}_a$, c_j and $\overline{\lambda}$. Substituting the Maxwell-Dirac equation (6.142) into the ratio equation (6.120), then solving for $\overline{k}_a$ gives

$$\overline{k}_a = \pm \overline{j}_a \sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1)]}, \tag{6.147}$$

which is only real within our allowed solution domain, where $c_k < 0$. Since $\overline{k}_b = c_k/\overline{k}_a$, we also have

$$\overline{k}_b = \mp (c_j/\overline{j}_a) \sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1)]}. \tag{6.148}$$

The other two parameters are

$$\overline{\sigma} = \pm 2\sqrt{c_j}, \tag{6.149}$$

$$\overline{k}_c = \overline{\lambda}^3 c_j \mp 2\overline{\lambda}\sqrt{c_j}. \tag{6.150}$$

The form of the four-vectors in the case where $\overline{k}_d = 0$, but with general $\overline{\lambda} > 0$ is

$$\overline{j}^\mu = \begin{pmatrix} \overline{j}_a e^{-2\overline{y}/\overline{\lambda}} + (c_j/\overline{j}_a) e^{2\overline{y}/\overline{\lambda}} \\ 0 \\ 0 \\ \overline{j}_a e^{-2\overline{y}/\overline{\lambda}} - (c_j/\overline{j}_a) e^{2\overline{y}/\overline{\lambda}} \end{pmatrix}, \tag{6.151}$$

$$\overline{k}^{\mu} = \begin{pmatrix} \pm\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1)]}[\overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} - (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}] \\ \overline{\lambda}^3 c_j \mp 2\overline{\lambda}\sqrt{c_j} \\ 0 \\ \pm\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1)]}[\overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} + (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}] \end{pmatrix}, \tag{6.152}$$

where the valid domain of c_j, for a given choice of $\overline{\lambda}$ and sign of $\overline{\sigma}$ are as summarized above. The field strength tensor components are

$$\overline{F}_a = -\frac{\overline{\lambda}\overline{j}_a}{2}, \tag{6.153}$$

$$\overline{F}_b = -\frac{\overline{\lambda}c_j}{2\overline{j}_a}, \tag{6.154}$$

which give electric and magnetic fields of the form

$$\overline{E}_y = -(\overline{\lambda}\overline{j}_a/2)\mathrm{e}^{-2\overline{y}/\overline{\lambda}} + (\overline{\lambda}c_j/2\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}, \tag{6.155}$$

$$\overline{M}_x = (\overline{\lambda}\overline{j}_a/2)\mathrm{e}^{-2\overline{y}/\overline{\lambda}} + (\overline{\lambda}c_j/2\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}. \tag{6.156}$$

6.3.5 Case with general $\overline{\lambda}$ and $\overline{k}_d$

We now allow both $\overline{\lambda}$ and $\overline{k}_d$ to assume arbitrary values, within limits to be defined shortly. The general Maxwell-Dirac equation for trans-boost symmetry is

$$c_k = \overline{\lambda}^6 c_j^2/4 \mp \overline{\lambda}^4 c_j^{3/2} + (\overline{\lambda}^2 - 1)c_j + \overline{k}_d^2/4, \tag{6.157}$$

which when combined with the ratio equation (6.120), requires that for real $\overline{\sigma}$, $c_j > 0$ and $c_k < 0$. The condition $\overline{\lambda} > 0$ still holds, as always.

In this general case, we assume the following hierarchy for fixing the parameters in this system, although other points of view are certainly possible. Initially, we choose our $\overline{\lambda}$ value, which must be > 0, but < 1 if the sign of $\overline{\sigma}$ is negative. Next, we choose c_j, which must lie within the ranges given in (6.145) for $\overline{\sigma} > 0$, or (6.146) for $\overline{\sigma} < 0$. These ranges are the same as for the $\overline{k}_d = 0$ case, and are a direct result of choosing the value of c_j before the $\overline{k}_d$.

If we were to reverse the order, and imposed a value of $\overline{k}_d$ first, then the $c_k = 0$ contours in Figure 6.13 would move in such a way as to make the allowed c_j range more restrictive than in (6.145) and (6.146). Indeed, Figure 6.12 shows this c_j "domain shrinking" for the case where $\overline{\lambda} = 1$. For that case when $\overline{k}_d = 0$, the c_j range is at its maximum extent, but as $\overline{k}_d$ increases, the c_j domain shrinks, until it disappears altogether at $\overline{k}_d = 3\sqrt{3}$. Such domain shrinking occurs for all $\overline{\lambda}$ values.

From the point of view of choosing the c_j first, the limit on the allowed values of $\overline{k}_d$ is obtained by setting $c_k = 0$ in (6.157), and solving for $\overline{k}_d$. This gives us an expression for the limiting value of $\overline{k}_d$, given a chosen set $\{\mathrm{sgn}(\overline{\sigma}), \overline{\lambda}, c_j\}$:

$$|\overline{k}_d| < 2\sqrt{-[\overline{\lambda}^6 c_j^2/4 \mp \overline{\lambda}^4 c_j^{3/2} + (\overline{\lambda}^2 - 1)c_j]} = |\overline{k}_{d,\max}|. \tag{6.158}$$

The four-vector parameters in terms of the set $\{\mathrm{sgn}(\overline{\sigma}), \overline{\lambda}, c_j, \overline{k}_d, \overline{j}_a\}$ are

$$\overline{k}_a = \pm\overline{j}_a\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1) + \overline{k}_d^2/4c_j]}, \tag{6.159}$$

$$\overline{k}_b = \mp(c_j/\overline{j}_a)\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1) + \overline{k}_d^2/4c_j]}, \tag{6.160}$$

and $\overline{\sigma}$ and $\overline{k}_c$ are the same as in (6.149) and (6.150). The general form of the four-vectors is therefore

$$\overline{j}^\mu = \begin{pmatrix} \overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} + (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}} \\ 0 \\ 0 \\ \overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} - (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}} \end{pmatrix}, \tag{6.161}$$

$$\overline{k}^\mu = \begin{pmatrix} \pm\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1) + \overline{k}_d^2/4c_j]}[\overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} - (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}] \\ \overline{\lambda}^3 c_j \mp 2\overline{\lambda}\sqrt{c_j} \\ \overline{k}_d \\ \pm\sqrt{-[\overline{\lambda}^6 c_j/4 \mp \overline{\lambda}^4 c_j^{1/2} + (\overline{\lambda}^2 - 1) + \overline{k}_d^2/4c_j]}[\overline{j}_a \mathrm{e}^{-2\overline{y}/\overline{\lambda}} + (c_j/\overline{j}_a)\mathrm{e}^{2\overline{y}/\overline{\lambda}}] \end{pmatrix}, \tag{6.162}$$

and the field strength constants, electric, and magnetic fields are the same as they are in (6.153)-(6.156).

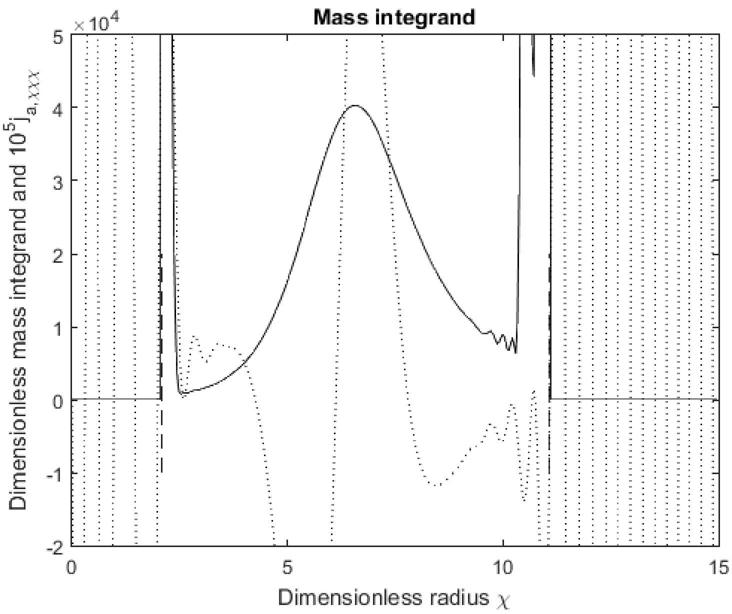

Figure 6.7: Comparison of $\chi^2\overline{T}_{00}$ (solid line) with a $10^5\times$ vertically exaggerated $\overline{j}_{a,\chi\chi\chi}$ plot (dotted line). The outer boundaries of the integration region, where $\chi\overline{\sigma} \approx 0$, are indicated by the vertical dashed lines. Numerical instability of $\overline{j}_{a,\chi\chi\chi}$ near the edges of the integrated region is apparent. The size of the largest solid spike is $\sim 10^9$.

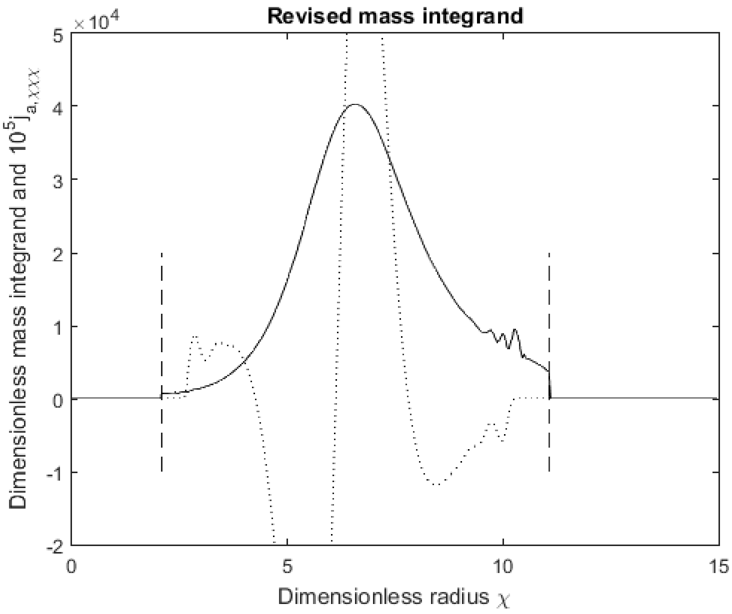

Figure 6.8: Comparison of $\chi^2\overline{T}_{00}$ (solid line) with a $10^5\times$ vertically exaggerated $\overline{j}_{a,\chi\chi\chi}$ plot (dotted line). The outer boundaries of the integration region, where $\chi\overline{\sigma} \approx 0$, are indicated by the vertical dashed lines. Numerical instability of $\overline{j}_{a,\chi\chi\chi}$ near the edges of the integrated region has been eliminated by setting the derivatives to zero at arbitrary points near the edges. This alteration removes the spikes entirely.

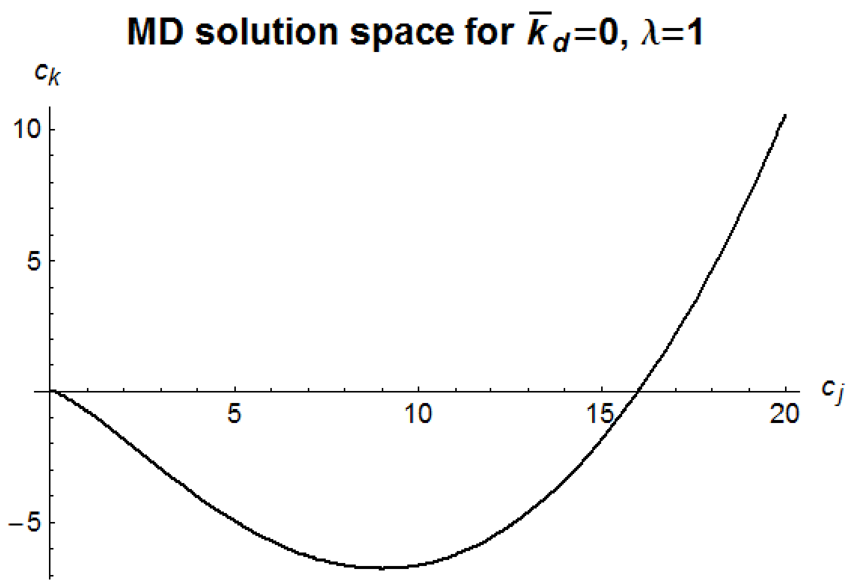

Figure 6.9: A plot of c_k vs. c_j for the case where $\overline{\lambda} = 1$, $\overline{k}_d = 0$.

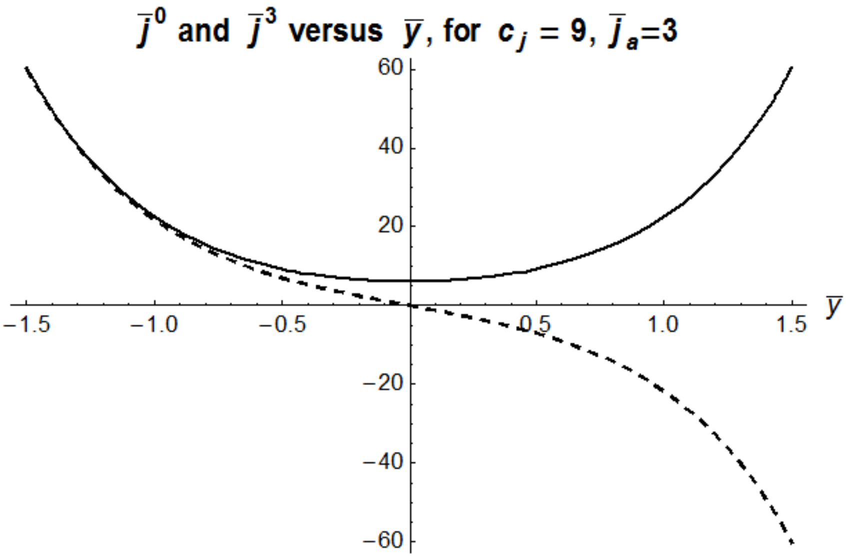

Figure 6.10: A plot of $\bar{j}^0$ (solid) and $\bar{j}^3$ (dashed) vs. $\bar{y}$ for the case where $\bar{\lambda} = 1$, $\bar{k}_d = 0$, $c_j = 9$ and $\bar{j}_a = 3$.

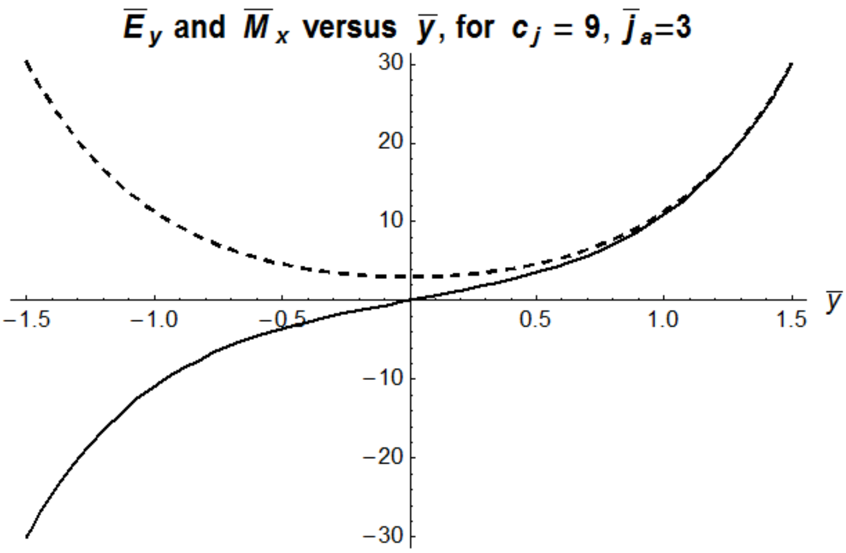

Figure 6.11: A plot of $\bar{E}_y$ (solid) and $\bar{M}_x$ (dashed) vs. $\bar{y}$ for the case where $\bar{\lambda} = 1$, $\bar{k}_d = 0$, $c_j = 9$ and $\bar{j}_a = 3$.

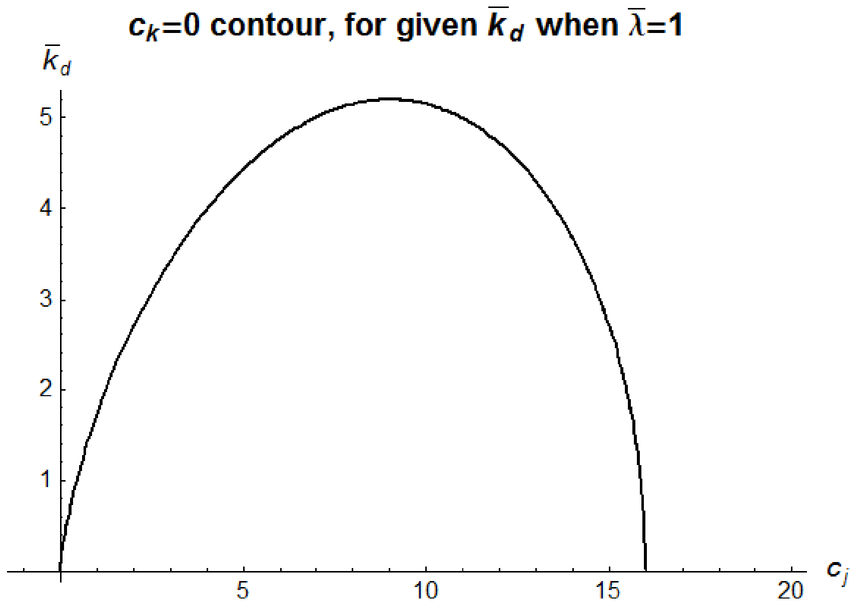

Figure 6.12: A plot of the $c_k = 0$ contour when $\overline{\lambda} = 1$, for given c_j and $\overline{k}_d$ values. $c_k < 0$ values lie underneath the contour. Note that $\overline{k}_d$ can take negative values, and this graph is symmetric about the horizontal axis.

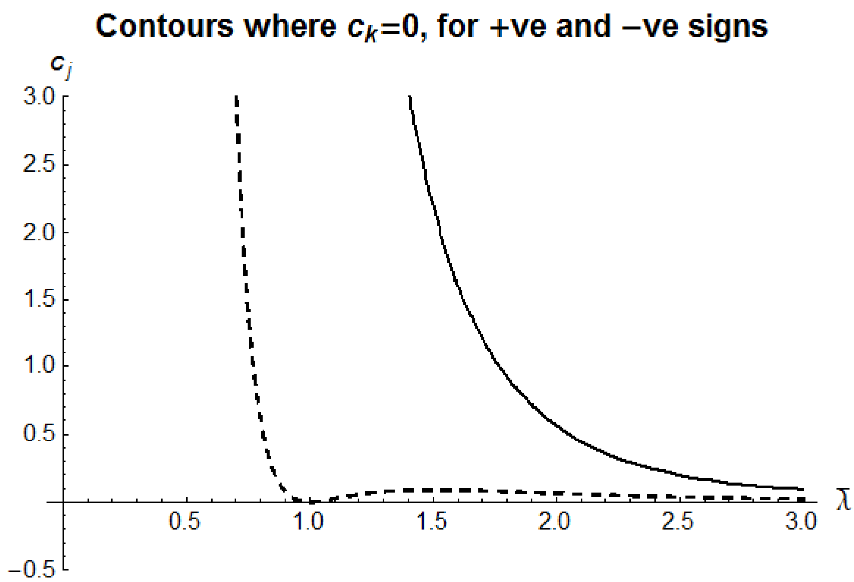

Figure 6.13: A plot of the $c_k = 0$ contours as functions of c_j and $\overline{\lambda}$ for the case where $\overline{k}_d = 0$. For the +ve sign Maxwell-Dirac equation ($\sigma < 0$), parameters are restricted to lie to the bottom-left of the dashed curve. For the -ve sign case ($\sigma > 0$), parameters are restricted to lie between the dotted and solid lines for $\overline{\lambda} > 1$, and anywhere under the solid line for $\overline{\lambda} < 1$.

Chapter 7

Generalization to Non-Abelian Gauge Fields

Before we draw this thesis to a close, we give a brief introduction to how the theory developed up to this point for the Abelian gauge field case can be extended to non-Abelian gauge fields. In particular, we consider the case where the Dirac spinors have a new *doublet* degree of freedom, so that they are of the form $\Psi_{i\alpha}$, where $i = 1, 2$ is the doublet index, and $\alpha = 1, 2, 3, 4$ is the usual Dirac spinor index for the given doublet component. The free particle Dirac equation for such a doublet spinor is the same as (1.1), but with ψ replaced by Ψ, and the terms in the parentheses being understood to be multiplied by the 2×2 identity matrix. According to the gauge principle, we can make this equation covariant under a local $SU(2)$ rotation in the doublet space, by replacing the derivative with the appropriate covariant derivative $D_\mu \equiv \partial_\mu + (ig/2)\boldsymbol{\tau} \cdot \not{\boldsymbol{W}}$.

Now, from the Abelian inversion case discussed in section 2.1, we required the charge conjugated Dirac equation for the algebraic manipulations leading to the inversion. This is also true in the current non-Abelian case, but in a more general way, such that an extra "conjugation" step involving the Pauli matrices acting on the doublet space is required, leading to an isospin-charge (or *isocharge*) conjugate spinor Ψ^{ic}. Note that this step is possible due to a convenient property of the Pauli matrices, which is not shared by the Gell-Mann matrices of the adjoint representation of the $SU(3)$ gauge group. This complicates the prospects of finding a similar algebraic inversion of the Dirac equation for the gauge field of QCD, but does not necessarily rule it out.

As in the Abelian case, we find that when pre-multiplying the isocharge conjugated and unconjugated Dirac equations by spinor terms like $\overline{\Psi}\tau_a\gamma_\mu$ in order to form doublet bilinears, we obtain expressions that can *almost* be inverted by dividing through by a scalar term, but not quite. In the Abelian case, this scalar term is $\sigma^2 - \omega^2$, however the non-Abelian analogue is a *matrix* with free Pauli and Dirac indices that can in principle be inverted using a Neumann series. The invertible matrix contains the rank-2 doublet bilinears $S_{a\mu\nu}$ and ${}^*S_{a\mu\nu}$, but we show how, in close analogy with the Abelian cases (2.13) and (2.14), these bilinears can be eliminated using non-Abelian Fierz identities, and replaced by bilinears of lower rank. These new Fierz identities are interesting in their own right, but they have the potential of simplifying explicit calculations of the inverted non-Abelian Dirac equation. Such explicit calculations are left for future work.

7.1 Non-Abelian SU(2) case

The $SU(2)$ gauge covariant Dirac equation for a doublet spinor Ψ is

$$[\mathrm{i}\partial\!\!\!/ - (g/2)\boldsymbol{\tau}\cdot \boldsymbol{W}\!\!\!\!/ - m]\Psi = 0 \tag{7.1}$$

where τ_a $(a = 1, 2, 3)$ are the non-commutative generators of infinitesimal rotations in doublet space and $W_{a\mu}$ are the Yang-Mills fields, the $SU(2)$ gauge fields analogous to A_μ. Explicitly, τ_a are the Pauli matrices, which obey the commutation relations

$$[\tau_a/2, \tau_b/2] = \mathrm{i}\epsilon_{ab}{}^{c}\tau_c/2 \tag{7.2}$$

where ϵ_{abc} is the rank-3 Levi-Civita tensor, and is antisymmetric under exchange of any two indices. Note that throughout this chapter, we use the Einstein summation convention for the Pauli indices, where a repeated index implies summation, with raising/lowering used to highlight the fact. Now, to derive the charge conjugate of this equation, we must follow the same process involved in charge conjugating the Abelian Dirac equation. The goal in the Abelian case was to flip the sign of the charge relative to all the other terms, and was achieved by complex conjugating the entire equation, then multiplying it by an invertible matrix U, such that $U\gamma_\mu^* U^{-1} = -\gamma_\mu$ and $U\psi^* = \psi^{\mathrm{c}}$. We follow the same process in the $SU(2)$ case, but not with the presupposition that the sign of g will necessarily be flipped. Complex conjugating and rearranging (7.1),

$$\{[\mathrm{i}\partial_\nu + (g/2)\boldsymbol{\tau}^{\mathrm{T}}\cdot \boldsymbol{W}_\nu]\gamma^{\nu *} + m\}\Psi^* = 0. \tag{7.3}$$

We have used the fact that $\boldsymbol{\tau}$ is Hermitian $(\tau_a^* = \tau_a^T)$. Multiply (7.3) by $I \otimes U$, with I being the 2×2 identity in doublet space, indicating that U commutes with $\boldsymbol{\tau}$ and acts only on the Dirac spinor degree of freedom

$$\begin{aligned} & 0 = \{[\mathrm{i}\partial_\nu + (g/2)\boldsymbol{\tau}^{\mathrm{T}}\cdot \boldsymbol{W}_\nu]U\gamma^{\nu *}U^{-1} + m\}U\Psi^* \\ \Rightarrow\ & 0 = \{[\mathrm{i}\partial_\nu + (g/2)\boldsymbol{\tau}^{\mathrm{T}}\cdot \boldsymbol{W}_\nu]\gamma^{\nu} - m\}\Psi^{\mathrm{c}}. \end{aligned} \tag{7.4}$$

To make the form of the gauge potential covariant, another step is required which is not present in the Abelian case, to convert the $\boldsymbol{\tau}^{\mathrm{T}}$ back to $\boldsymbol{\tau}$. This is done by multiplying (7.4) by $\epsilon \equiv \mathrm{i}\tau_2$, such that we make use of the Pauli identity $\tau_a = -\epsilon\tau_a^{\mathrm{T}}\epsilon^{-1}$,

$$\begin{aligned} & 0 = \{[\mathrm{i}\partial_\nu + (g/2)\epsilon\boldsymbol{\tau}^{\mathrm{T}}\epsilon^{-1}\cdot \boldsymbol{W}_\nu]\gamma^{\nu} - m\}\epsilon\Psi^{\mathrm{c}} \\ \Rightarrow\ & 0 = [\mathrm{i}\partial\!\!\!/ - (g/2)\boldsymbol{\tau}\cdot \boldsymbol{W}\!\!\!\!/ - m]\Psi^{\mathrm{ic}} \end{aligned} \tag{7.5}$$

where we have defined $\epsilon\Psi^{\mathrm{c}} \equiv \Psi^{\mathrm{ic}}$ as the isospin-charge conjugate (henceforth, IC) spinor. Note that the sign of the coupling constant g has reverted back, so that

(7.5) is of exactly the same form as (7.1). Mimicking the Abelian case (2.2), we rearrange (7.1) and (7.5) into the more convenient forms

$$\gamma^\nu \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu \Psi \equiv \tau^b \gamma^\nu \Psi W_{b\nu} = \Phi \tag{7.6}$$

$$\gamma^\nu \boldsymbol{\tau} \cdot \boldsymbol{W}_\nu \Psi^{\mathrm{ic}} \equiv \tau^b \gamma^\nu \Psi^{\mathrm{ic}} W_{b\nu} = \Phi^{\mathrm{ic}} \tag{7.7}$$

where $\Phi \equiv 2g^{-1}(\mathrm{i}\partial\!\!\!/ - m)\Psi$, $\Phi^{\mathrm{ic}} \equiv 2g^{-1}(\mathrm{i}\partial\!\!\!/ - m)\Psi^{\mathrm{ic}}$ and we have replaced the triplet vector dot-product notion with a sum over Pauli components b. Multiplying equation (7.6) by $\overline{\Psi}\tau_a\gamma_\mu$, applying the Dirac identity (A.10), as well as the Pauli identity (A.28),

$$\tau_a \tau^b W_{b\mu} = (\delta_a{}^b + \mathrm{i}\epsilon_a{}^{bc}\tau_c) W_{b\mu} = W_{a\mu} + \mathrm{i}\epsilon_a{}^{bc}\tau_c W_{b\mu} \tag{7.8}$$

we obtain the expression

$$\overline{\Psi}\Psi W_{a\mu} - \mathrm{i}\overline{\Psi}\sigma_\mu{}^\nu \Psi W_{a\nu} + \mathrm{i}\epsilon_a{}^{bc}\overline{\Psi}\tau_c\Psi W_{b\mu} + \epsilon_a{}^{bc}\overline{\Psi}\tau_c\sigma_\mu{}^\nu\Psi W_{b\nu} = \overline{\Psi}\tau_a\gamma_\mu\Phi \tag{7.9}$$

with the form of the IC equation being exactly the same. This is similar to (2.6), but with "extra" terms contracted with the Levi-Civita symbol on the left-hand side. We require a non-Abelian analogue of the bilinear relationship (A.35), which is

$$\overline{\Psi}^{\mathrm{ic}}(\tau_i \otimes \Gamma)\Psi^{\mathrm{ic}} = -\overline{\Psi}(\epsilon^{-1}\tau_i{}^{\mathrm{T}}\epsilon) \otimes (C^{-1}\Gamma^{\mathrm{T}}C)\Psi \tag{7.10}$$

with Γ being an element of the Dirac-Clifford algebra as before, and τ_i an element of the Pauli algebra, where $i = 0, 1, 2, 3$ and $i = 0$ corresponds to the 2×2 identity. Explicit sign relations for particular values of $(\tau_i \otimes \Gamma)$ are not given here, although they can easily be calculated following the same method as in (A.35). Following the same method as with the Abelian case, we *subtract* the IC of (7.9) from (7.9), apply the appropriate current sign relationships from (7.10), then rearrange to get

$$\overline{\Psi}\Psi g W_{a\mu} + \epsilon_a{}^{bc}\overline{\Psi}\tau_c\sigma_\mu{}^\nu\Psi g W_{b\nu} = \mathrm{i}(\overline{\Psi}\tau_a\gamma_\mu\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\gamma_\mu\Psi) - 2m\overline{\Psi}\tau_a\gamma_\mu\Psi. \tag{7.11}$$

We will now define a suite of non-Abelian currents, that will be used throughout the rest of this chapter:

$$J_i = \overline{\Psi}\tau_i\Psi \tag{7.12a}$$

$$J_{i\mu} = \overline{\Psi}\tau_i\gamma_\mu\Psi \tag{7.12b}$$

$$S_{i\mu\nu} = \overline{\Psi}\tau_i\sigma_{\mu\nu}\Psi \tag{7.12c}$$

$${}^*S_{i\mu\nu} = \overline{\Psi}\tau_i\gamma_5\sigma_{\mu\nu}\Psi \tag{7.12d}$$

$$K_{i\mu} = \overline{\Psi}\tau_i\gamma_5\gamma_\mu\Psi \tag{7.12e}$$

$$K_i = \overline{\Psi}\tau_i\gamma_5\Psi. \tag{7.12f}$$

There are 64 current densities altogether, excluding ${}^*S_{i\mu\nu}$ from the count, since

$${}^*S_{i\mu\nu} = (\mathrm{i}/2)\epsilon_{\mu\nu\sigma\rho}S_i{}^{\sigma\rho}. \tag{7.13}$$

However, a consequence of the results presented in the next section is that the number of linearly independent current densities is lower than 64, as the $S_{i\mu\nu}$ terms in any expression can be eliminated entirely. Equation (7.11) can now be rewritten in the more compact form

$$\left(J_0 \delta_\mu{}^\nu \delta_a{}^b - \epsilon_a{}^{cb} S_{c\mu}{}^\nu\right) g W_{b\nu} = \mathrm{i}(\overline{\Psi}\tau_a\gamma_\mu \partial\!\!\!/ \Psi - \overline{\Psi} \overleftarrow{\partial\!\!\!/}\, \tau_a \gamma_\mu \Psi) - 2m J_{a\mu}. \quad (7.14)$$

It is apparent that there is an additional matrix term on the left-hand side of (7.11) that is not present in the Abelian case, which prevents us from immediately finding an inverted form for $W_{a\mu}$. Now, let us consider, as with the Abelian case, an alternative formulation of the above expression, by pre-multiplying (7.6) and (7.7) by $\overline{\Psi}\tau_a\gamma_5\gamma_\mu$. Applying the sign relationships via (7.10) and *subtracting* the IC equation from the non-IC equation gives

$$\left(K_0 \delta_\mu{}^\nu \delta_a{}^b - \epsilon_a{}^{cb*} S_{c\mu}{}^\nu\right) g W_{b\nu} = \mathrm{i}(\overline{\Psi}\tau_a\gamma_5\gamma_\mu \partial\!\!\!/ \Psi + \overline{\Psi} \overleftarrow{\partial\!\!\!/}\, \tau_a \gamma_5\gamma_\mu \Psi), \quad (7.15)$$

in which the mass term vanishes, as in the Abelian case. We could go a step further and *add* the two equations (7.14) and (7.15), then divide by the scalar terms to give

$$\left[\delta_\mu{}^\nu \delta_a{}^b - \frac{\epsilon_a{}^{cb}(S_{c\mu}{}^\nu + {}^*S_{c\mu}{}^\nu)}{J_0 + K_0}\right] W_{b\nu}$$
$$= \frac{1}{g}\frac{\mathrm{i}(\overline{\Psi}\tau_a\gamma_\mu \partial\!\!\!/ \Psi + \overline{\Psi}\tau_a\gamma_5\gamma_\mu \partial\!\!\!/ \Psi - \overline{\Psi} \overleftarrow{\partial\!\!\!/}\, \tau_a\gamma_\mu\Psi + \overline{\Psi} \overleftarrow{\partial\!\!\!/}\, \tau_a\gamma_5\gamma_\mu\Psi) - 2mJ_{a\mu}}{J_0 + K_0}. \quad (7.16)$$

The matrix on the left-hand side is of the form $(I - N)$, which is invertible by way of the Neumann series

$$(I - N)^{-1} = \sum_{n=0}^{\infty} N^n = I + N + N^2 + ... \quad (7.17)$$

for $N \in \mathbb{C}^{n\times n}$, which has condition of convergence $\rho(N) < 1$, where $\rho(N)$ is the spectral radius of N [24]. In the following section, we will show that the terms in the expansion of $(I - N)^{-1}$ can be converted from contractions involving spin-current tensors to contractions involving J, K Lorentz scalar and vector currents exclusively, by deriving appropriate Fierz identities. Firstly, a brief comment on some of the various other objects which the Dirac and IC Dirac equations, (7.6) and (7.7), can be multiplied by to form bilinear-field coupling expressions. We could multiply either of these equations on the left by objects of the form $\overline{\Psi}\tau_i\Gamma$, where Γ is an irreducible element of the Dirac-Clifford basis (including the dual of the rank-2 tensor, $\gamma_5\sigma_{\mu\nu}$), then either add or subtract the resulting equations. For example, consider the case where $\Gamma = I$, $i = 0$, then we multiply (7.6) and (7.7) by $\overline{\Psi}$ and $\overline{\Psi}^{\mathrm{ic}}$ respectively. Subtracting the IC equation from the non-IC equation gives

$$(g/2)J^{b\nu}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\,\Psi) - mJ_0, \quad (7.18)$$

and adding them gives

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\,\Psi = -\overline{\Psi}\partial\!\!\!/\Psi. \quad (7.19)$$

We could choose to eliminate the bilinear with the left-acting derivative operator by substituting the second equation into the first, resulting in the expression

$$(g/2)J^{b\nu}W_{b\nu} = \mathrm{i}\overline{\Psi}\partial\!\!\!/\Psi - mJ_0. \quad (7.20)$$

This equation describes the coupling of the Lorentz vector current $J_{a\mu}$ with the vector potential field via contraction of both the Pauli and Lorentz indices, resulting in a sum of Lorentz and Pauli scalar terms. Unlike the invertible cases discussed above, this equation has no free indices and therefore can not be inverted via multiplication by an appropriate matrix. Since these types of equations may provide valuable information in future studies, all $\overline{\Psi}\tau_i\Gamma$ multiplication options are listed in appendix B.

7.2 Non-Abelian Fierz identities

Consider the 8×8 matrix formed by the product $\Psi\overline{\Psi}$. In the $SU(2)$ doublet degree of freedom, this is a 2×2 matrix, and in the Dirac spinor degree of freedom, it is 4×4 matrix. In the pure Dirac case, the product of two Dirac spinors can be expanded via a Fierz expansion (2.9). There are 16 terms in the sum, and the coefficients a_R are the Dirac bilinears $\overline{\chi}\Gamma_R\psi$, multiplied by a numerical constant, with Γ_R being the Rth element of the sixteen-component basis of the Dirac-Clifford algebra. It is interesting to note that extensions of Fierz expansions such as this have been described in arbitrary higher dimensions in [12]. Now, in the pure 2×2 case, the Fierz expansion for the matrix formed by the product of two doublet spinors v is

$$vv^\dagger = \sum_{i=0}^{3} c_i\tau_i = (1/2)(v^\dagger v)\tau_0 + (1/2)(v^\dagger\tau_a v)\tau^a \tag{7.21}$$

with $a = 1, 2, 3$, to make four terms in the sum in total. As discussed before, τ_a are the Pauli matrices, and τ_0 is the 2×2 identity. Coefficients c_i are pure $SU(2)$ isospin bilinears $v^\dagger\tau_i v$. Now, the basis of the $\Psi\overline{\Psi}$ Fierz expansion is the tensor product of the Dirac and Pauli bases

$$\begin{aligned}\Psi\overline{\Psi} &= \sum_{i=0}^{3}\sum_{R=1}^{16} a_{Ri}(\Gamma_R\otimes\tau_i) = (1/8)J_i(I\otimes\tau^i) + (1/8)J_{i\mu}(\gamma^\mu\otimes\tau^i)\\ &+(1/16)S_{i\mu\nu}(\sigma^{\mu\nu}\otimes\tau^i) - (1/8)K_{i\mu}(\gamma_5\gamma^\mu\otimes\tau^i) + (1/8)K_i(\gamma_5\otimes\tau^i)\end{aligned} \tag{7.22}$$

where the coefficients are the $SU(2)$ bilinears as previously defined, and are derived by pre-multiplying (7.22) by an element of $(\Gamma_R\otimes\tau_i)$, utilizing trace identities, and then solving for the leftover a_{Ri}. Henceforth, we will exclude the tensor product symbol explicitly, however its presence is implied in any product of Dirac and Pauli matrices.

The Fierz expansion can used to expand products of non-Abelian currents in terms of other currents in the Dirac-Pauli algebra, for example

$$\begin{aligned}J_a{}^\mu K_b{}^\nu &= \overline{\Psi}\tau_a\gamma^\mu(\Psi\overline{\Psi})\tau_b\gamma_5\gamma^\nu\Psi\\ &= -(1/8)J_i\overline{\Psi}\gamma_5\gamma^\mu\gamma^\nu\tau_a\tau^i\tau_b\Psi + (1/8)J_{i\sigma}\overline{\Psi}\gamma_5\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\tau_b\Psi\\ &\quad -(1/16)S_{i\sigma\epsilon}\overline{\Psi}\gamma_5\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu\tau_a\tau^i\tau_b\Psi\\ &\quad +(1/8)K_{i\sigma}\overline{\Psi}\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\tau_b\Psi + (1/8)K_i\overline{\Psi}\gamma^\mu\gamma^\nu\tau_a\tau^i\tau_b\Psi\end{aligned} \tag{7.23}$$

which, after a lengthy expansion, converting all of the Dirac and Pauli matrix products to sums of irreducible terms gives

$$\begin{aligned}
J_a{}^\mu K_b{}^\nu = (1/4)[&\mathrm{i}J_a{}^*S_b{}^{\mu\nu} + \mathrm{i}J_b{}^*S_a{}^{\mu\nu} - \mathrm{i}K_a S_b{}^{\mu\nu} - \mathrm{i}K_b S_a{}^{\mu\nu} + J_a{}^\mu K_b{}^\nu \\
&+ J_a{}^\nu K_b{}^\mu + J_b{}^\mu K_a{}^\nu + J_b{}^\nu K_a{}^\mu - J_{a\sigma}K_b{}^\sigma \eta^{\mu\nu} - J_{b\sigma}K_a{}^\sigma \eta^{\mu\nu} \\
&+ \delta_{ab}(\mathrm{i}J_0{}^*S_0{}^{\mu\nu} - \mathrm{i}J_c{}^*S^{c\mu\nu} - \mathrm{i}K_0 S_0{}^{\mu\nu} + \mathrm{i}K_c S^{c\mu\nu} + J_0{}^\mu K_0{}^\nu \\
&+ J_0{}^\nu K_0{}^\mu - J_c{}^\mu K^{c\nu} - J_c{}^\nu K^{c\mu} - J_{0\sigma}K_0{}^\sigma \eta^{\mu\nu} + J_{c\sigma}K^{c\sigma}\eta^{\mu\nu})] \\
&+ (1/4)\epsilon_{ab}{}^c[-\mathrm{i}J_0 K_c \eta^{\mu\nu} + \mathrm{i}K_0 J_c \eta^{\mu\nu} + J_{c\sigma}J_{0\lambda}\epsilon^{\mu\nu\sigma\lambda} + K_{c\sigma}K_{0\lambda}\epsilon^{\mu\nu\sigma\lambda} \\
&+ (1/2)\mathrm{i}(-S_0{}^\mu{}_\sigma{}^*S_c{}^{\sigma\nu} - S_0{}^\nu{}_\sigma{}^*S_c{}^{\sigma\mu} + S_c{}^\mu{}_\sigma{}^*S_0{}^{\sigma\nu} + S_c{}^\nu{}_\sigma{}^*S_0{}^{\sigma\mu})]. \qquad (7.24)
\end{aligned}$$

In the set of Fierz identities for JK vector products, we call this the $a-b$ case. There are three other cases of vector current products: $J_a{}^\mu K_0{}^\nu$, $J_0{}^\mu K_a{}^\nu$ and $J_0{}^\mu K_0{}^\nu$, which we call the $a-0$, $0-a$ and $0-0$ cases respectively. Due to their long-winded nature, we leave the derivation and listing of these Fierz identities to appendix F. Now, if we wish to express the inverted form of the vector potential equation (7.16) independently of the rank-2 spin current (skew) tensor $S_i{}^{\mu\nu}$ and its dual, we must obtain an identity to describe $S_i{}^{\mu\nu}$ solely in terms of J, K scalar and vector current densities. Since $S_i{}^{\mu\nu}$ is antisymmetric under exchange of its two Lorentz indices, we should form antisymmetric terms from the two-vector product $J_a{}^\mu K_b{}^\nu$, then Fierz expand using (7.22) and solve for $S_0{}^{\mu\nu}$, $S_a{}^{\mu\nu}$, ${}^*S_0{}^{\mu\nu}$ or ${}^*S_a{}^{\mu\nu}$.

Let us define the suite of 16 *Abelian* currents as follows

$$\sigma = \overline{\psi}\psi \qquad (7.25a)$$

$$j_\mu = \overline{\psi}\gamma_\mu\psi \qquad (7.25b)$$

$$s_{\mu\nu} = \overline{\psi}\sigma_{\mu\nu}\psi \qquad (7.25c)$$

$${}^*s_{\mu\nu} = \overline{\psi}\gamma_5\sigma_{\mu\nu}\psi \qquad (7.25d)$$

$$k_\mu = \overline{\psi}\gamma_5\gamma_\mu\psi \qquad (7.25e)$$

$$\omega = \overline{\psi}\gamma_5\psi \qquad (7.25f)$$

where the dual of the rank-2 skew tensor current may be calculated as in (7.13). If we consider the Fierz identity for the Abelian version of the rank-2 spin current tensor [10], [27],

$$s_{\mu\nu} = \frac{[\sigma\epsilon_{\mu\nu}{}^{\rho\kappa} - \mathrm{i}\omega(\delta_\mu{}^\rho\delta_\nu{}^\kappa - \delta_\mu{}^\kappa\delta_\nu{}^\rho)]j_\rho k_\kappa}{\sigma^2 - \omega^2} \qquad (7.26)$$

we can see that, in the non-Abelian case, we should consider Fierz expansions of antisymmetric current combinations with Lorentz structure of the form $J^\mu K^\nu - J^\nu K^\mu$ and $\epsilon^{\mu\nu\rho\kappa}J_\rho K_\kappa$. Due to the presence of the extra internal Pauli index, we need to take consideration of how this will vary the form of $S_i{}^{\mu\nu}$ compared with $s^{\mu\nu}$. As discussed in appendix G, the correct approach is to treat the derivation of $S_0{}^{\mu\nu}$ and $S_a{}^{\mu\nu}$ separately. For the $i=0$ case, we calculate the Fierz identities for $J_0{}^\mu K_0{}^\nu - J_a{}^\nu K^{a\mu}$, and $J_a{}^\mu K^{a\nu} - J_0{}^\nu K_0{}^\mu$, then add to form

$$J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu} = 2\mathrm{i}(J_0{}^*S_0{}^{\mu\nu} - K_0 S_0{}^{\mu\nu}). \qquad (7.27)$$

The antisymmetric part is calculated by adding $\epsilon^{\mu\nu\rho\kappa}J_{0\rho}K_{0\kappa}$ and $\epsilon^{\mu\nu\rho\kappa}J_{a\rho}K^a{}_\kappa$ to form

$$\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa = 2(J_0 S_0{}^{\mu\nu} - K_0{}^*S_0{}^{\mu\nu}). \qquad (7.28)$$

Taking the combination

$$\begin{aligned} &J_0\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa - \mathrm{i}K_0(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu}) \\ &\qquad = 2(J_0^2 S_0{}^{\mu\nu} - J_0K_0{}^*S_0{}^{\mu\nu} + J_0K_0{}^*S_0{}^{\mu\nu} - K_0^2 S_0{}^{\mu\nu}), \end{aligned} \tag{7.29}$$

the middle two terms on the right-hand side cancel, and we can rearrange to obtain the expression

$$S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[J_0\epsilon^{\mu\nu}{}_{\rho\kappa} - \mathrm{i}K_0(\delta_\rho{}^\mu\delta_\kappa{}^\nu - \delta_\rho{}^\nu\delta_\kappa{}^\mu)]J_i{}^\rho K^{i\kappa}. \tag{7.30}$$

The Abelian (7.26) and non-Abelian (7.30) cases share a close similarity, with the main differences being the factor of 1/2, and the sum over the internal Pauli index in the non-Abelian case. Similarly, we can switch the terms that J_0 and K_0 multiply in (7.29) to obtain an expression for the dual

$${}^*S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[K_0\epsilon^{\mu\nu}{}_{\rho\kappa} - \mathrm{i}J_0(\delta_\rho{}^\mu\delta_\kappa{}^\nu - \delta_\rho{}^\nu\delta_\kappa{}^\mu)]J_i{}^\rho K^{i\kappa}. \tag{7.31}$$

Note that we can check the validity of this dual identity by using the defining identity (7.13) on either (7.30) or (7.31). Now taking a step further, the scalar currents inside the square brackets can be eliminated by taking the sum or difference of (7.30) and (7.31)

$$S_0{}^{\mu\nu} \pm {}^*S_0{}^{\mu\nu} = (1/2)(J_0 \mp K_0)^{-1}[\epsilon^{\mu\nu}{}_{\rho\kappa} \mp \mathrm{i}(\delta_\rho{}^\mu\delta_\kappa{}^\nu - \delta_\rho{}^\nu\delta_\kappa{}^\mu)]J_i{}^\rho K^{i\kappa}. \tag{7.32}$$

Now, in order to form an expression for $S_a{}^{\mu\nu}$, we again need to consider antisymmetric combinations of JK vector current products, but in such a way as to have the Pauli vector triplet index $i = a = 1, 2, 3$ present in first order only, combined with the Pauli scalar singlet index $i = 0$. That is, we shall be dealing with the *rank-1* JK Pauli vector combinations, the $0 - a$ and $a - 0$ cases, as opposed to the $0 - 0$ *rank-0* or $a - b$ *rank-2* Pauli index cases. As discussed in appendix G, the appropriate Lorentz antisymmetric combinations are

$$\begin{aligned} &(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) - (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu) \\ &\qquad = \mathrm{i}J_0{}^*S_a{}^{\mu\nu} + \mathrm{i}J_a{}^*S_0{}^{\mu\nu} - \mathrm{i}K_0 S_a{}^{\mu\nu} - \mathrm{i}K_a S_0{}^{\mu\nu} \end{aligned} \tag{7.33}$$

as well as

$$\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) = J_0 S_a{}^{\mu\nu} + J_a S_0{}^{\mu\nu} - K_0{}^*S_a{}^{\mu\nu} - K_a{}^*S_0{}^{\mu\nu}. \tag{7.34}$$

Taking the combination

$$\begin{aligned} &J_0\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) - \mathrm{i}K_0[(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) - (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu)] \\ &\qquad = (J_0^2 - K_0^2)S_a{}^{\mu\nu} + (J_0J_a - K_0K_a)S_0{}^{\mu\nu} + (K_0J_a - J_0K_a){}^*S_0{}^{\mu\nu} \end{aligned} \tag{7.35}$$

we can see that after rearranging to solve for $S_a{}^{\mu\nu}$, we need to substitute the identities (7.30) and (7.31) to eliminate the Pauli singlet ($i = 0$) skew tensor current densities. The final expression is

$$S_a{}^{\mu\nu} = (J_0^2 - K_0^2)^{-1}[J_0\epsilon^{\mu\nu}{}_{\rho\kappa} - \mathrm{i}K_0(\delta_\rho{}^\mu\delta_\kappa{}^\nu - \delta_\rho{}^\nu\delta_\kappa{}^\mu)](J_a{}^\rho K_0{}^\kappa + J_0{}^\rho K_a{}^\kappa)$$

$$
\begin{aligned}
&- \frac{J_0^2 + K_0^2}{2(J_0^2 - K_0^2)^2}[J_a \epsilon^{\mu\nu}{}_{\rho\kappa} + \mathrm{i}K_a(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)] J_i{}^\rho K^{i\kappa} \\
&+ \frac{J_0 K_0}{(J_0^2 - K_0^2)^2}[K_a \epsilon^{\mu\nu}{}_{\rho\kappa} + \mathrm{i}J_a(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)] J_i{}^\rho K^{i\kappa}, \qquad (7.36)
\end{aligned}
$$

which bears less resemblance to the Abelain case (7.26) than the Pauli singlet case (7.30) does. To calculate the dual, we follow exactly the same process, but switch the J_0 and K_0 in (7.35), which after some rearrangement and substitution gives

$$
\begin{aligned}
{}^*S_a{}^{\mu\nu} = {}& (J_0^2 - K_0^2)^{-1}[K_0 \epsilon^{\mu\nu}{}_{\rho\kappa} - \mathrm{i}J_0(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)](J_a{}^\rho K_0{}^\kappa + J_0{}^\rho K_a{}^\kappa) \\
&+ \frac{J_0^2 + K_0^2}{2(J_0^2 - K_0^2)^2}[K_a \epsilon^{\mu\nu}{}_{\rho\kappa} + \mathrm{i}J_a(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)] J_i{}^\rho K^{i\kappa} \\
&- \frac{J_0 K_0}{(J_0^2 - K_0^2)^2}[J_a \epsilon^{\mu\nu}{}_{\rho\kappa} + \mathrm{i}K_a(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)] J_i{}^\rho K^{i\kappa}. \qquad (7.37)
\end{aligned}
$$

Again, this can be confirmed by using (7.13). As in the $i = 0$ case, taking the sum or difference of these two identities removes the current terms from inside the square brackets. Following some straightforward algebraic manipulation, we obtain a somewhat simpler form

$$
\begin{aligned}
S_a{}^{\mu\nu} \pm {}^*S_a{}^{\mu\nu} = {}& (1/2)(J_0 \mp K_0)^{-2}[\epsilon^{\mu\nu}{}_{\rho\kappa} \mp \mathrm{i}(\delta_\rho{}^\mu \delta_\kappa{}^\nu - \delta_\rho{}^\nu \delta_\kappa{}^\mu)] \\
&\cdot [2(J_0 \mp K_0)(J_a{}^\rho K_0{}^\kappa + J_0{}^\rho K_a{}^\kappa) - (J_a \mp K_a) J_i{}^\rho K^{i\kappa}]. \qquad (7.38)
\end{aligned}
$$

The relative simplicity of (7.38) compared with (7.36) and (7.37) is the reason we chose to pursue an invertible Dirac equation of the form (7.16) as opposed to (7.14) or (7.15) alone. In particular, the extra simplicity will have a profound impact on the complexity of the higher-power terms in the Neumann series form of the inverse matrix.

Chapter 8

Conclusions

In this thesis, we developed a manifestly gauge invariant tensor formalism for the Maxwell-Dirac equation, with the philosophical aim of describing the self-coupled relativistic quantum electrodynamics entirely in terms of *observables*. Once obtained, we then subjected this highly complex non-linear system of partial differential equations to symmetry restrictions, such that they be invariant under transformations by elements from specifically chosen subgroups of the Poincaré group. Following further theoretical development of the manifestly symmetric bilinear form of the stress-energy tensor, the spherically symmetric reduction was further restricted to be static. A single ODE was obtained, and numerical methods were applied to find two solutions corresponding to static shells of charge centred on the origin. The total mass of the simpler single-hump solution was obtained by applying the stress-energy tensor, reduced under this symmetry. Total charge was also calculated. The algebraic system resulting from the "trans-boost" reduction was also investigated, and hyperbolic closed-form solutions were found for a special case. Finally, the algebraic inversion of the Dirac equation was extended to the case where the gauge field is non-Abelian.

Initially, in chapter 2, a brief recap of the known Dirac equation inversion for an Abelian gauge field A^μ was given, followed by a short discussion on Fierz identities and bilinear products of spinors. Next, the inverted Dirac equation was recast into a purely tensorial form, and a gauge-independent vector potential B^μ was defined by subtracting the gauge-dependent parts from A^μ. Following a brief discussion on the tetrad of mutually orthogonal vector fields, the electromagnetic field strength tensor $F_{\mu\nu}$ was recast into a manifestly gauge invariant form, involving only B^μ and gauge independent tensors. The resulting set of tensorial, manifestly gauge invariant Maxwell-Dirac equations, Fierz identities and consistency conditions was summarized in the list (2.44)-(2.50).

In chapter 3, the invariants and the forms of four vector fields for a given set of generators were calculated by finding solutions of the vanishing Lie derivative PDEs via the method of characteristics. Four example Poincaré subalgebras were studied in particular: the standard spherical and cylindrical symmetries, as well as the more unusual splitting $P_{11,2}$ (screw) and non-splitting $\widetilde{P}_{13,10}$ (trans-boost) subalgebras of PWZ [35].

The first two subalgebras reduced dependent variables to functions of two independent variables, whereas the last two reduced dependent variables to constants, because of transitive action on Minkowski space [31]. The number of independent variables for the different symmetry subalgebras had a large impact on the complexity of the reduced Maxwell-Dirac system. For each subalgebra discussed in chapter 3, in chapter 4 the invariant forms were applied to the bilinear tensor fields σ, ω, j^μ and k^μ. The reduced forms of the Fierz identities, gauge invariant vector potential, field strength tensor, and Maxwell equations were calculated, with the reduced set of Maxwell-Dirac equations and consistency conditions presented at the end of each subsection.

In the case of spherical symmetry, the Maxwell-Dirac equations took the form of two coupled PDEs (4.36) and (4.37) in terms of the dependent functions j_a, j_b, σ and ω, as well as their t and r derivatives up to third order. Additional information is provided in the spherically reduced Fierz identity (4.38) and the continuity equations, (4.39) and (4.40). Interestingly, magnetic monopoles appear in the field strength tensor, but ultimately play no part in the coupled Maxwell-Dirac system.

Cylindrical symmetry resulted in a more complicated system, with the Maxwell-Dirac equations given implicitly by the set (4.62a), (4.62d), (4.63) and (4.64), where the field strength tensor dependent functions of t and ρ are given explicitly in appendix D. Five more equations are provided by the three Fierz identities (4.42) and (4.43), as well as the two continuity equations (4.65a) and (4.65b).

The first of the PWZ subalgebras, the splitting "screw" group $P_{11,2}$, resulted in a strong reduction of the Maxwell-Dirac system, to the point where the only allowed solution was the trivial one (4.73), where all of the tensor fields are zero. This provided a good demonstration of the fact that if a symmetry is too restrictive, the only solutions are vanishing fields.

The last example was the non-splitting "trans-boost" subalgebra $\widetilde{P}_{13,10}$, where the Maxwell-Dirac system, combined with the Fierz identities and continuity equations boiled down to the two algebraic equations (4.106) and (4.107); solutions correspond to finding sets of constants j_a, j_b, k_a, k_b and k_d (where one other than k_d can be eliminated) that solve this equation for a given type of symmetry, defined by the continuous parameter $\lambda > 0$.

In chapter 5, we demonstrated that bilinear forms for the stress-energy tensor ($\Theta_{\mu\nu}$ and $T_{\mu\nu}$ respectively) can indeed be calculated, by applying Fierz identities to the spinor terms appearing in the Belinfante and variational general relativistic calculational schemes. In the Belinfante case, the Fierz mapping was applied to the spinorial Belinfante tensor, and in the variational case, it was applied to the spinorial Lagrangian prior to the vierbein deformation. Despite the fact that these two methods are quite independent, they are in agreement in accordance with Goedecke's conjecture [20] and Lord's subsequent equivalence proof [33], lending extra weight to the validity of the bilinear representation of the Maxwell-Dirac system.

However, there is a point of view from which this automatic agreement is somewhat surprising. When taking into consideration the details of the functional Jacobian corresponding to the spinor to bilinear mapping, one would expect there to be extra constraint terms entering into the bilinearized Lagrangian, with the lack of such terms in (5.32) leading to a disagreement with the Belinfante tensor in the *bilinear* representation. A transcription of spinor electrodynamics into gauge invariant quantities in this spirit, has been given in the functional formalism by Rudolph and Kijowski [28], [29]. In their bosonic transcription, Green's functions are given as functional integrals in whose integrands there are always additional accompanying field-dependent factors, and so an effective bosonic, local, purely Lagrangian formulation is not obtained. The details of the agreement between $\Theta_{\mu\nu}$ and $T_{\mu\nu}$ for the bilinear case, although highly encouraging, remains a matter deserving of further study.

Putting these technical concerns aside, we then turned to an example to demonstrate how the bilinear stress-energy tensor is reduced under spherical symmetry, using the generic $SO(3)$ invariant forms for scalar and four-vector fields discussed in chapters 3 and 4. We found that the stress-energy components could be described in terms of three functions (5.93)-(5.95) corresponding to the interacting Dirac matter contribution, as well as a single function (5.92), corresponding to the energy density of the Maxwell field.

Chapter 6 represents the culmination of the previous theoretical development, whereby we took two example Poincaré symmetry subgroup reductions of the bilinearized Maxwell-Dirac equations, and obtained solutions for each case. The first symmetry reduction we dealt with was for static spherical symmetry in section 6.1, which implied that the only non-zero component of the current four-vector j^μ was the $\mu = 0$ component j_a, corresponding to charge density. Following non-dimensionalization, after taking into account the Fierz identity (6.2) and the partial conservation of axial current (6.3), we obtained the single non-linear fourth-order ODE for the dimensionless charge density (6.31). A brief analytical consideration revealed that, subject to the special constraint (6.33) which from (6.5) corresponds to the case where $\overline{\sigma} = 0$, there is an *exact solution* (6.37). Due to a singularity at the origin caused by the χ^{-2} proportionality, this solution is not a physically realistic one. However at large values of χ the exponential part dominates, so for the negative sign case with asymptotic exponential decay, (6.37) appears to be consistent with Radford's theorem [38], requiring that stationary spinor (static bilinear) solutions to the Maxwell-Dirac equations are strictly localized and decay exponentially.

Linearizing about this solution revealed no further information since a first-order perturbation results in the same form for the exact solution (6.37), but with a different arbitrary constant. By setting all of the derivatives in (6.31) to zero, we found only one physically sensible equilibrium point, corresponding to $\overline{j}_{a,e} = 0$, the first-order perturbation about which yielded the weakly non-linear ODE for J (6.50), which closely resembled the full ODE (6.31), but lacked the square root term. This ODE admits a slightly more general class of exponential exact solutions (6.51), where the power ± 2 is replaced by an arbitrary constant B.

The vanishing square root term turns out to be a convenient feature because, in lieu of an extra constraint condition imposing that $\overline{j}_a$ be such that the argument of the square root term is strictly positive, numerical calculations would tend to have large imaginary parts. Further work on (large amplitude) solutions to the fully non-linear system should include such a constraint. Converting the problem from one of calculating solutions J to (6.50), to the algebraic one of finding a set of Fourier coefficients b_n that minimize a set of residuals R_n representing the ODE according to (6.53), we found after some experimentation with Gaussian-form initial guesses, the two solutions in Figure 6.5. The solution forms for weakly non-linear and fully non-linear ODEs, J and $\bar{j}_a$ respectively, closely resemble one another when the J solution form is bootstrapped as the initial guess for a $\bar{j}_a$ solution calculation. There is some slight difference between the solutions corresponding to the two sign cases on the square root term in (6.31), as can be seen in Figure 6.3, but for smaller distributions, as in the double hump case in Figure 6.4, the difference is much smaller. This is attributable to the contribution of the square root term vanishing as $\bar{j}_a \to 0$, where the weakly non-linear equation dominates. In light of the following total mass-energy and charge calculations, we find that for physically reasonable mass values the dimensionless amplitude of the charge density $\bar{j}_a$ must be at an order of magnitude such that only the weakly non-linear part needs to be considered. The charge quantization constraint (6.91) should also be taken into account.

Following the calculation and non-dimensionalization of the static, spherically symmetric stress-energy tensor in section 6.2, we proceeded to calculate the total charge and mass corresponding to the single-hump solution in Figure 6.3. Since the solutions for each sign case are very similar, we chose the positive sign case to work with. However, due to the nasty behaviour of derivative terms at the edges of the $\bar{j}_a$ distribution, the integrand of the mass integral (6.86) produced large spikes in this region. Taming these numerical instabilities by manually setting the derivative terms to zero just before the fluctuations became too serious, resulted in an elimination of the spikes, as in Figure 6.8. The total mass for the single-hump $\bar{j}_a$ distribution with the mass integrand spikes eliminated was calculated to be $M \approx 7.3 \times 10^4$ MeV (or 73 GeV), with total charge $Q \approx 4.4 \times 10^4 e$. Since these values are so large, the single hump solution corresponding to a static shell of charge should be considered as primarily of theoretical interest.

Next, in section 6.3, we consider the algebraic form of the Maxwell-Dirac system resulting from the $\widetilde{P}_{13,10}$ "trans-boost" symmetry reduction, and how solutions can be obtained from this relatively easily. Following non-dimensionalization, we proceeded initially by taking the simplifying assumptions $\bar{\lambda} = 1$ and $\bar{k}_d = 0$, causing two of the terms from (6.110) to vanish. Defining the two new parameters $c_j = \bar{j}_a \bar{j}_b$ and $c_k = \bar{k}_a \bar{k}_b$, we found that for all of our physical fields to be real, that the values of these parameters are restricted to $c_j > 0$ and $c_k < 0$, which only holds in the finite range $0 < c_j < 16$. The range of c_j corresponds to the $c_k = 0$ contour. For the special case where $c_j = 9$ and $\bar{j}_a = 3$, the components of $\bar{j}^\mu$ and the electric and magnetic fields are given by the respective hyperbolic forms (6.134) and (6.135), shown in Figures 6.10 and 6.11. These solutions represent sheets of charge in the $x - z$ plane undergoing laminar flow in the z-direction, with flux density magnitude as a function of y. Since the physical parameters diverge to infinite values as $y \to \infty$, these solutions should be considered to be primarily of theoretical interest as closed-form solutions to the Maxwell-Dirac system. The hyperbolic solution forms were first obtained by Legg [31], but from our work here, it is clear that they correspond to a special case.

Extending to the case where general values of $\bar{k}_d$ are allowed (but $\bar{\lambda} = 1$) had the effect of shrinking the allowed "solution domain" as $|\overline{k}_d|$ increased, from the maximum range $0 < c_j < 16$ corresponding to $\overline{k}_d = 0$, to the single point $c_j = 9$ at the maximum allowed value $|\overline{k}_d| = 3\sqrt{3}$. This variation of the allowed c_j range as a function of $|\overline{k}_d|$ is shown in Figure 6.12. In a similar, but more complicated way, allowing $\overline{\lambda}$ to vary freely but setting $\overline{k}_d = 0$, resulted in expressions for the variation of the allowed c_j range, as a function of $\overline{\lambda}$ and the sign of $\overline{\sigma}$. These ranges are given in (6.145) and (6.146), with the corresponding $c_k = 0$ contours shown in Figure 6.13. Lastly, allowing the $\overline{k}_d$ parameter to also vary freely, caused the allowed c_j range to shrink for all $\overline{\lambda}$ values, as expected. From the point of view of setting $\overline{\lambda}$ first, the range of $\overline{k}_d$ is restricted by the values of $\overline{\lambda}$ and c_j, as in (6.158).

Finally, in chapter 7, the inversion of the Dirac equation to solve for the vector potential was extended to the non-Abelian $SU(2)$ gauge field. An extension of the charge conjugation operation was made in order to make the form of the gauge field generators covariant, and the algebraic system tractable. In analogy to previous studies performed on the Abelian case, the non-Abelian Dirac system was re-written in terms of doublet bilinear current densities, by way of multiplication by the terms $\overline{\Psi}\tau_a\gamma_\mu$ and $\overline{\Psi}\tau_a\gamma_5\gamma_\mu$. Combining these equations, an invertible form was achieved, provided we made use of a Neumann expansion to describe the form of the inverse matrix. In order to eliminate the rank-2 skew tensor current densities from the Neumann expansion, we were motivated to derive appropriate Fierz identities by considering antisymmetric combinations of $J_{i\mu}K_{j\nu}$ current density products. Expressions for $S_{0\mu\nu}$, ${}^*S_{0\mu\nu}$, $S_{a\mu\nu}$ and ${}^*S_{a\mu\nu}$ were subsequently derived, and the convenient linear combinations $S_{0\mu\nu} \pm {}^*S_{0\mu\nu}$ and $S_{a\mu\nu} \pm {}^*S_{a\mu\nu}$ were formed.

Some options for further work include describing explicitly the conditions for the convergence of the inverse matrix Neumann series, as well as the affect of Poincaré symmetry reductions. As with the Abelian case, a broader study of the non-Abelian Fierz identities is also in order, in particular obtaining a complete minimal set, from which all other redundant Fierz identities can be derived. The scope of the inversion may also be extended to $SU(2) \times U(1)$ and $SU(3)$ gauge fields. In the latter case, the lack of an extended analogue of the $SU(2)$ isospin-charge conjugate spinors Ψ^{ic} suggests that the analysis of Fierz identities will have to confront a yet more involved set of doublet bilinear current densities, and the Dirac equation, a more complicated inversion calculation.

Appendix A

Algebraic Identities

A.1 General Identities

Throughout this thesis we use the Levi-Civita symbol, defined as

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ -1 & \text{if is an odd permutation} \\ 0 & \text{otherwise,} \end{cases} \tag{A.1}$$

with the additional property

$$\epsilon_{\mu\nu\rho\sigma} = \det(\eta)\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}. \tag{A.2}$$

For convenience, we another rank-4 antisymmetric symbol, the shorthand for which is

$$\delta^{\mu\nu\rho\sigma} = \mathrm{i}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{A.3}$$

A.2 Dirac Identities

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.4}$$

$$[\gamma^\mu, \gamma^\nu] = -2\mathrm{i}\sigma^{\mu\nu} \tag{A.5}$$

$$\gamma^5 = \gamma_5 = -(\mathrm{i}/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \tag{A.6}$$

$$\gamma_5^2 = I \tag{A.7}$$

$$\{\gamma_5, \gamma^\mu\} = 0 \tag{A.8}$$

$$[\gamma_5, \sigma^{\mu\nu}] = 0 \tag{A.9}$$

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu} \tag{A.10}$$

$$\gamma^\mu\gamma_\mu = 4 \tag{A.11}$$

$$\gamma^\mu\gamma_5\gamma_\mu = -4\gamma_5 \tag{A.12}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda = \eta^{\mu\nu}\gamma^\lambda + \eta^{\nu\lambda}\gamma^\mu - \eta^{\mu\lambda}\gamma^\nu - \mathrm{i}\epsilon^{\mu\nu\lambda\sigma}\gamma_5\gamma_\sigma \tag{A.13}$$

$$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu \tag{A.14}$$
$$\gamma^\nu\gamma_5\gamma^\mu\gamma_\nu = 2\gamma_5\gamma^\mu \tag{A.15}$$
$$\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\epsilon = \eta^{\mu\nu}\eta^{\sigma\epsilon} + \eta^{\nu\sigma}\eta^{\mu\epsilon} - \eta^{\mu\sigma}\eta^{\nu\epsilon} - \mathrm{i}\eta^{\mu\nu}\sigma^{\sigma\epsilon} - \mathrm{i}\eta^{\nu\sigma}\sigma^{\mu\epsilon} + \mathrm{i}\eta^{\mu\sigma}\sigma^{\nu\epsilon} + \mathrm{i}\eta^{\mu\epsilon}\sigma^{\sigma\nu} + \mathrm{i}\eta^{\nu\epsilon}\sigma^{\mu\sigma} + \mathrm{i}\eta^{\sigma\epsilon}\sigma^{\nu\mu} - \mathrm{i}\epsilon^{\mu\nu\sigma\epsilon}\gamma_5 \tag{A.16}$$
$$\gamma^\epsilon\sigma^{\mu\nu} = \mathrm{i}\eta^{\epsilon\mu}\gamma^\nu - \mathrm{i}\eta^{\epsilon\nu}\gamma^\mu + \epsilon^{\mu\nu\epsilon\sigma}\gamma_5\gamma_\sigma \tag{A.17}$$
$$\sigma^{\mu\nu}\gamma^\epsilon = \mathrm{i}\eta^{\nu\epsilon}\gamma^\mu - \mathrm{i}\eta^{\mu\epsilon}\gamma^\nu + \epsilon^{\mu\nu\epsilon\sigma}\gamma_5\gamma_\sigma \tag{A.18}$$
$$\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu = \mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 + \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda \tag{A.19}$$
$$\gamma^\sigma\sigma^{\mu\nu}\gamma_\sigma = 0 \tag{A.20}$$
$$\sigma^{\mu\nu}\gamma_\mu = -3\mathrm{i}\gamma^\nu, \tag{A.21}$$
$$\sigma^{\mu\nu}\gamma^\rho\gamma_\mu = 3\mathrm{i}\eta^{\nu\rho} + \sigma^{\nu\rho}, \tag{A.22}$$
$$\sigma^{\mu\nu}\sigma^{\rho\tau}\gamma_\mu = \eta^{\nu\rho}\gamma^\tau - \eta^{\nu\tau}\gamma^\rho + \mathrm{i}\epsilon^{\nu\rho\tau\sigma}\gamma_5\gamma_\sigma, \tag{A.23}$$
$$\gamma^\mu\sigma_{\nu\mu} = -3\mathrm{i}\gamma_\nu, \tag{A.24}$$
$$\gamma^\mu\gamma^\rho\sigma_{\nu\mu} = 3\mathrm{i}\delta_\nu{}^\rho - \sigma_\nu{}^\rho, \tag{A.25}$$
$$\gamma^\mu\sigma^{\rho\tau}\sigma_{\nu\mu} = \delta_\nu{}^\tau\gamma^\rho - \delta_\nu{}^\rho\gamma^\tau + \mathrm{i}\eta_{\nu\kappa}\epsilon^{\kappa\rho\tau\sigma}\gamma_5\gamma_\sigma, \tag{A.26}$$
$$-\epsilon^{\lambda\rho\sigma\epsilon}\epsilon_\lambda{}^{\mu\nu\tau} = \eta^{\rho\mu}\eta^{\sigma\nu}\eta^{\epsilon\tau} - \eta^{\rho\mu}\eta^{\epsilon\nu}\eta^{\sigma\tau} + \eta^{\rho\nu}\eta^{\sigma\tau}\eta^{\epsilon\mu} - \eta^{\rho\nu}\eta^{\epsilon\tau}\eta^{\sigma\mu} + \eta^{\rho\tau}\eta^{\sigma\mu}\eta^{\epsilon\nu} - \eta^{\rho\tau}\eta^{\epsilon\mu}\eta^{\sigma\nu} \tag{A.27}$$

A.3 Pauli Identities

$$\tau_a\tau_b = \delta_{ab} + \mathrm{i}\epsilon_{abd}\tau^d \tag{A.28}$$
$$\tau_a\tau_c\tau_b = \tau_a\delta_{bc} + \tau_b\delta_{ac} - \tau_c\delta_{ab} - \mathrm{i}\epsilon_{abc} \tag{A.29}$$

A.4 Charge conjugation identities

$$C^{-1}\gamma_\mu^{\mathrm{T}}C = -\gamma_\mu \tag{A.30}$$
$$C^{-1}\gamma_5^{\mathrm{T}}C = \gamma_5 \tag{A.31}$$
$$C^{-1}\sigma_{\mu\nu}^{\mathrm{T}}C = \sigma_{\mu\nu} \tag{A.32}$$
$$C^{-1}(\gamma_\mu\gamma_\nu)^{\mathrm{T}}C = \gamma_\nu\gamma_\mu \tag{A.33}$$
$$C^{-1}(\gamma_5\gamma_\mu)^{\mathrm{T}}C = \gamma_5\gamma_\mu \tag{A.34}$$

The relationship between a charge conjugate bilinear and a regular bilinear, for commuting spinor fields is

$$\overline{\psi}^{\mathrm{c}}\Gamma\chi^{\mathrm{c}} = -\overline{\chi}C^{-1}\Gamma^{\mathrm{T}}C\psi, \tag{A.35}$$

where Γ is an element of the Dirac-Clifford algebra. Some particular examples, using the above charge conjugation identities are

$$\overline{\psi}^{\mathrm{c}}\psi^{\mathrm{c}} = -\overline{\psi}\psi \tag{A.36}$$

$$\overline{\psi}^{\mathrm{c}}\gamma_\mu\psi^{\mathrm{c}} = \overline{\psi}\gamma_\mu\psi \tag{A.37}$$

$$\overline{\psi}^{\mathrm{c}}\sigma_{\mu\nu}\psi^{\mathrm{c}} = \overline{\psi}\sigma_{\mu\nu}\psi \tag{A.38}$$

$$\overline{\psi}^{\mathrm{c}}\gamma_\mu\gamma^\nu(\partial_\nu\psi^{\mathrm{c}}) = -(\partial_\nu\overline{\psi})\gamma^\nu\gamma_\mu\psi \tag{A.39}$$

$$\overline{\psi}^{\mathrm{c}}\psi = \overline{\psi}^{\mathrm{c}}\gamma_5\gamma_\mu\psi = \overline{\psi}^{\mathrm{c}}\gamma_5\psi = 0. \tag{A.40}$$

The last identity is due to the fact these expressions equal their own negatives.

A.5 Dirac bilinear notation

The shorthand for gauge-independent Dirac bilinear tensors is as follows:

$$\sigma = \overline{\psi}\psi \tag{A.41}$$

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{A.42}$$

$$s^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi \tag{A.43}$$

$$^*s^{\mu\nu} = \overline{\psi}\gamma_5\sigma^{\mu\nu}\psi \tag{A.44}$$

$$k^\mu = \overline{\psi}\gamma_5\gamma^\mu\psi \tag{A.45}$$

$$\omega = \overline{\psi}\gamma_5\psi. \tag{A.46}$$

With regards to convention, note that some authors define the pseudoscalar bilinear to be $\omega \equiv \overline{\psi}\mathrm{i}\gamma_5\psi$. The gauge-dependent bilinear tensors are

$$m^\mu + \mathrm{i}n^\mu = \overline{\psi}^{\mathrm{c}}\gamma^\mu\psi \tag{A.47}$$

$$m^\mu = \mathrm{Re}[\overline{\psi}^{\mathrm{c}}\gamma^\mu\psi] = (1/2)(\overline{\psi}^{\mathrm{c}}\gamma^\mu\psi + \overline{\psi}\gamma^\mu\psi^{\mathrm{c}}) \tag{A.48}$$

$$n^\mu = \mathrm{Im}[\overline{\psi}^{\mathrm{c}}\gamma^\mu\psi] = (\mathrm{i}/2)(\overline{\psi}\gamma^\mu\psi^{\mathrm{c}} - \overline{\psi}^{\mathrm{c}}\gamma^\mu\psi). \tag{A.49}$$

The last two equations follow from the bilinear complex conjugation identity

$$(\overline{\chi}\Gamma\psi)^* = \overline{\psi}(\gamma_0\Gamma^\dagger\gamma_0)\chi, \tag{A.50}$$

which implies, for $\Gamma = \gamma^\mu$

$$(\overline{\psi}^{\mathrm{c}}\gamma^\mu\psi)^* = \overline{\psi}\gamma^\mu\psi^{\mathrm{c}}. \tag{A.51}$$

Appendix B

Expressions from bilinearization of the Dirac equation

B.1 Abelian case

We list here the various expressions resulting from left-multiplying the Dirac equation and its charge conjugate by $\overline{\psi}\Gamma$ and $\overline{\psi}^{\mathrm{c}}\Gamma$ respectively (where Γ is an element of the Dirac-Clifford algebra), then 1) subtracting the charge conjugate equation from the regular equation and 2) adding the charge conjugate equation to the regular equation.

$\Gamma = \overline{\psi}$:

$$j^\nu A_\nu = \frac{\mathrm{i}}{2q}[\overline{\psi}\gamma^\nu(\partial_\nu\psi) - (\partial_\nu\overline{\psi})\gamma^\nu\psi] - \frac{m\sigma}{q} \tag{B.1}$$

$$\partial_\nu j^\nu = 0 \tag{B.2}$$

$\Gamma = \overline{\psi}\gamma_5$:

$$\partial_\nu k^\nu = -2\mathrm{i}m\omega \tag{B.3}$$

$$k^\nu A_\nu = \frac{\mathrm{i}}{2q}[\overline{\psi}\gamma_5\gamma^\nu(\partial_\nu\psi) - (\partial_\nu\overline{\psi})\gamma_5\gamma^\nu\psi] \tag{B.4}$$

$\Gamma = \overline{\psi}\gamma_\mu$:

$$s_\mu{}^\nu A_\nu = \frac{\mathrm{i}}{2q}[\overline{\psi}\sigma_\mu{}^\nu(\partial_\nu\psi) - (\partial_\nu\overline{\psi})\sigma_\mu{}^\nu\psi] - \frac{\partial_\mu\sigma}{2q} \tag{B.5}$$

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] + \partial_\nu s_\mu{}^\nu - 2mj_\mu}{\sigma} \tag{B.6}$$

$\Gamma = \overline{\psi}\gamma_5\gamma_\mu$:

$$^*s_\mu{}^\nu A_\nu = \frac{\mathrm{i}}{2q}[\overline{\psi}\gamma_5\sigma_\mu{}^\nu(\partial_\nu\psi) - (\partial_\nu\overline{\psi})\gamma_5\sigma_\mu{}^\nu\psi] - \frac{\partial_\mu\omega}{2q} - \frac{\mathrm{i}mk_\mu}{q} \tag{B.7}$$

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] + \partial_\nu {}^*s_\mu{}^\nu}{\omega} \tag{B.8}$$

$\Gamma = \overline{\psi}\sigma_{\mu\nu}$:

$$\delta_{\mu\nu}{}^{\rho\sigma}A_\rho j_\sigma = \frac{1}{2q}\left\{\mathrm{i}\delta_{\mu\nu}{}^{\rho\sigma}[\overline{\psi}\gamma_\sigma(\partial_\rho\psi) - (\partial_\rho\overline{\psi})\gamma_\sigma\psi] - \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}\partial_\rho k_\sigma\right\} \tag{B.9}$$

$$\epsilon_{\mu\nu}{}^{\rho\sigma}A_\rho k_\sigma = \frac{1}{2q}\left\{\mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}[\overline{\psi}\gamma_5\gamma_\sigma(\partial_\rho\psi) - (\partial_\rho\overline{\psi})\gamma_5\gamma_\sigma\psi] - \mathrm{i}\delta_{\mu\nu}{}^{\rho\sigma}\partial_\rho j_\sigma - 2ms_{\mu\nu}\right\} \tag{B.10}$$

$\Gamma = \overline{\psi}\gamma_5\sigma_{\mu\nu}$:

$$\epsilon_{\mu\nu}{}^{\rho\sigma}A_\rho j_\sigma = \frac{1}{2q}\left\{\mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}[\overline{\psi}\gamma_\sigma(\partial_\rho\psi) - (\partial_\rho\overline{\psi})\gamma_\sigma\psi] - \mathrm{i}\delta_{\mu\nu}{}^{\rho\sigma}\partial_\rho k_\sigma\right\} \tag{B.11}$$

$$\delta_{\mu\nu}{}^{\rho\sigma}A_\rho k_\sigma = \frac{1}{2q}\left\{\mathrm{i}\delta_{\mu\nu}{}^{\rho\sigma}[\overline{\psi}\gamma_5\gamma_\sigma(\partial_\rho\psi) - (\partial_\rho\overline{\psi})\gamma_5\gamma_\sigma\psi] - \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}\partial_\rho j_\sigma + 2m^*s_{\mu\nu}\right\} \tag{B.12}$$

B.2 Non-Abelian case

Here we list the system of expressions that result when we multiply the $SU(2)$ gauge covariant Dirac equation and its isospin-charge conjugate equation (IC equation),

$$\tau^b\gamma^\nu W_{b\nu}\Psi = (2/g)(\mathrm{i}\partial\!\!\!/ - m)\Psi, \tag{B.13}$$

$$\tau^b\gamma^\nu W_{b\nu}\Psi^{\mathrm{ic}} = (2/g)(\mathrm{i}\partial\!\!\!/ - m)\Psi^{\mathrm{ic}}, \tag{B.14}$$

from the left by a matrix of the general form $\overline{\Psi}\tau_i\Gamma$. For each of these matrices, we will obtain three equations, the order of the list being: (a) subtract IC from non-IC equation, (b) add IC from non-IC equation, and (c) the result obtained by combining (a) and (b) to eliminate the bilinear with left-acting derivative operator, $\overleftarrow{\partial\!\!\!/}$. Note that the equation we use for substitution will we written in terms of the aforementioned $\overleftarrow{\partial\!\!\!/}$ bilinear.

Multiply by $\overline{\Psi}$:

$$(g/2)J^{b\nu}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\Psi) - mJ_0 \tag{B.15}$$

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\Psi = -\overline{\Psi}\partial\!\!\!/\Psi \tag{B.16}$$

$$(g/2)J^{b\nu}W_{b\nu} = \mathrm{i}\overline{\Psi}\partial\!\!\!/\Psi - mJ_0 \tag{B.17}$$

Multiply by $\overline{\Psi}\tau_a$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\Psi = g\epsilon_a{}^{bc}J_c{}^\nu W_{b\nu} - \overline{\Psi}\tau_a\partial\!\!\!/\Psi \tag{B.18}$$

$$(g/2)J_0{}^\nu W_{a\nu} = (\mathrm{i}/2)(\overline{\Psi}\tau_a\partial\!\!\!/\Psi - \Psi\overleftarrow{\partial\!\!\!/}\tau_a\Psi) - mJ_a \tag{B.19}$$

$$(g/2)(J_0{}^{\nu}\delta_a{}^b + \mathrm{i}J_c{}^{\nu}\epsilon_a{}^{bc})W_{b\nu} = \mathrm{i}\overline{\Psi}\tau_a\partial\!\!\!/\Psi - mJ_a \tag{B.20}$$

Multiply by $\overline{\Psi}\gamma_5$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\Psi = \overline{\Psi}\gamma_5\partial\!\!\!/\Psi + 2imK_0 \tag{B.21}$$

$$(g/2)K^{b\nu}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\gamma_5\partial\!\!\!/\Psi + \overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\Psi) \tag{B.22}$$

$$(g/2)K^{b\nu}W_{b\nu} = \mathrm{i}\overline{\Psi}\gamma_5\partial\!\!\!/\Psi - mK_0 \tag{B.23}$$

Multiply by $\overline{\Psi}\tau_a\gamma_5$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\tau_a\Psi = -g\mathrm{i}K_0{}^{\nu}W_{a\nu} - \overline{\Psi}\gamma_5\tau_a\partial\!\!\!/\Psi \tag{B.24}$$

$$\mathrm{i}(g/2)\epsilon_a{}^{bc}K_c{}^{\nu}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\gamma_5\tau_a\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\tau_a\Psi) - mK_a \tag{B.25}$$

$$(g/2)(K_0{}^{\nu}\delta_a{}^b + \mathrm{i}K_c{}^{\nu}\epsilon_a{}^{bc})W_{b\nu} = \mathrm{i}\overline{\Psi}\gamma_5\tau_a\partial\!\!\!/\Psi - mK_a \tag{B.26}$$

Multiply by $\overline{\Psi}\gamma_\mu$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_\mu\Psi = -\overline{\Psi}\gamma_\mu\partial\!\!\!/\Psi - gS^b{}_\mu{}^\nu W_{b\nu} \tag{B.27}$$

$$(g/2)J^bW_{b\mu} = (\mathrm{i}/2)(\overline{\Psi}\gamma_\mu\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_\mu\Psi) - mJ_{0\mu} \tag{B.28}$$

$$(g/2)(J^b\delta_\mu{}^\nu - \mathrm{i}S^b{}_\mu{}^\nu)W_{b\nu} = \mathrm{i}\overline{\Psi}\gamma_\mu\partial\!\!\!/\Psi - mJ_{0\mu} \tag{B.29}$$

Multiply by $\overline{\Psi}\tau_a\gamma_\mu$:

$$(g/2)(J_0\delta_\mu{}^\nu\delta_a{}^b + S_{c\mu}{}^\nu\epsilon_a{}^{bc})W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\tau_a\gamma_\mu\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\gamma_\mu\Psi) - mJ_{a\mu} \tag{B.30}$$

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\gamma_\mu\Psi = g(J_c\delta_\mu{}^\nu\epsilon_a{}^{bc} - S_{0\mu}{}^\nu\delta_a{}^b)W_{b\mu} - \overline{\Psi}\tau_a\gamma_\mu\partial\!\!\!/\Psi \tag{B.31}$$

$$(g/2)(J_0\delta_\mu{}^\nu\delta_a{}^b + \mathrm{i}J_c\delta_\mu{}^\nu\epsilon_a{}^{bc} - \mathrm{i}S_{0\mu}{}^\nu\delta_a{}^b + S_{c\mu}{}^\nu\epsilon_a{}^{bc})W_{b\nu} = \mathrm{i}\overline{\Psi}\tau_a\gamma_\mu\partial\!\!\!/\Psi - mJ_{a\mu} \tag{B.32}$$

Multiply by $\overline{\Psi}\gamma_5\gamma_\mu$:

$$(g/2)\mathrm{i}^*S^b{}_\mu{}^\nu W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\gamma_\mu\Psi - \overline{\Psi}\gamma_5\gamma_\mu\partial\!\!\!/\Psi) + mK_{0\mu} \tag{B.33}$$

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\gamma_\mu\Psi = -g\mathrm{i}K^bW_{b\mu} - \overline{\Psi}\gamma_5\gamma_\mu\partial\!\!\!/\Psi \tag{B.34}$$

$$(g/2)(K^b\delta_\mu{}^\nu - \mathrm{i}^*S^b{}_\mu{}^\nu)W_{b\nu} = \mathrm{i}\overline{\Psi}\gamma_5\gamma_\mu\partial\!\!\!/\Psi - mK_{0\mu} \tag{B.35}$$

Multiply by $\overline{\Psi}\tau_a\gamma_5\gamma_\mu$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\gamma_5\gamma_\mu\Psi = -\mathrm{i}g(K_0\delta_a{}^b\delta_\mu{}^\nu + {}^*S_{c\mu}{}^\nu\epsilon_a{}^{bc})W_{b\nu} - \overline{\Psi}\tau_a\gamma_5\gamma_\mu\partial\!\!\!/\Psi \tag{B.36}$$

$$\mathrm{i}(g/2)(K_c\delta_\mu{}^\nu\epsilon_a{}^{bc} - {}^*S_{0\mu}{}^\nu\delta_a{}^b)W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\tau_a\gamma_5\gamma_\mu\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\tau_a\gamma_5\gamma_\mu\Psi) - mK_{a\mu} \tag{B.37}$$

$$(g/2)(K_0\delta_\mu{}^\nu\delta_a{}^b + \mathrm{i}K_c\delta_\mu{}^\nu\epsilon_a{}^{bc} - \mathrm{i}^*S_{0\mu}{}^\nu\delta_a{}^b + {}^*S_{c\mu}{}^\nu\epsilon_a{}^{bc})W_{b\nu} = \mathrm{i}\overline{\Psi}\tau_a\gamma_5\gamma_\mu\partial\!\!\!/\Psi - mK_{a\mu} \tag{B.38}$$

Multiply by $\overline{\Psi}\sigma_{\rho\epsilon}$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\sigma_{\rho\epsilon}\Psi = \overline{\Psi}\sigma_{\rho\epsilon}\partial\!\!\!/\Psi - (\delta_\epsilon{}^\nu\delta_\rho{}^\sigma - \delta_\rho{}^\nu\delta_\epsilon{}^\sigma)gJ^b{}_\sigma W_{b\nu} \tag{B.39}$$

$$(g/2)\epsilon_{\rho\epsilon}{}^{\nu\sigma}K^{b}{}_{\sigma}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\sigma_{\rho\epsilon}\partial\!\!\!/\Psi + \overline{\Psi}\overleftarrow{\partial\!\!\!/}\sigma_{\rho\epsilon}\Psi) - mS_{0\rho\epsilon} \tag{B.40}$$

$$(g/2)[\epsilon_{\rho\epsilon}{}^{\nu\sigma}K^{b}{}_{\sigma} + \mathrm{i}(\delta_{\epsilon}{}^{\nu}\delta_{\rho}{}^{\sigma} - \delta_{\rho}{}^{\nu}\delta_{\epsilon}{}^{\sigma})J^{b}{}_{\sigma}]W_{b\nu} = \mathrm{i}\overline{\Psi}\sigma_{\rho\epsilon}\partial\!\!\!/\Psi - mS_{0\rho\epsilon} \tag{B.41}$$

Multiply by $\overline{\Psi}\tau_a\sigma_{\rho\epsilon}$:

$$\begin{aligned}(g/2)[\epsilon_{\rho\epsilon}{}^{\nu\sigma}K_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})\mathrm{i}\epsilon_{a}{}^{bc}J_{c\sigma}]W_{b\nu}\\ = (\mathrm{i}/2)(\overline{\Psi}\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\sigma_{\rho\epsilon}\tau_a\Psi) - mS_{a\rho\epsilon}\end{aligned} \tag{B.42}$$

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\sigma_{\rho\epsilon}\tau_a\Psi = g[(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})J_{0\sigma}\delta_{a}{}^{b} + \epsilon_{\rho\epsilon}{}^{\nu\sigma}\epsilon_{a}{}^{bc}K_{c\sigma}]W_{b\nu} - \overline{\Psi}\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi \tag{B.43}$$

$$\begin{aligned}(g/2)[\epsilon_{\rho\epsilon}{}^{\nu\sigma}(K_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}K_{c\sigma}\epsilon_{a}{}^{bc}) + \mathrm{i}(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})(J_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}J_{c\sigma}\epsilon_{a}{}^{bc})]W_{b\nu}\\ = \mathrm{i}\overline{\Psi}\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi - mS_{a\rho\epsilon}\end{aligned} \tag{B.44}$$

Multiply by $\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}$:

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\sigma_{\rho\epsilon}\Psi = -\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\partial\!\!\!/\Psi - \mathrm{i}\epsilon_{\rho\epsilon}{}^{\nu\sigma}gJ^{b}{}_{\sigma}W_{b\nu} \tag{B.45}$$

$$(g/2)\mathrm{i}(\delta_{\epsilon}{}^{\nu}\delta_{\rho}{}^{\sigma} - \delta_{\rho}{}^{\nu}\delta_{\epsilon}{}^{\sigma})K^{b}{}_{\sigma}W_{b\nu} = (\mathrm{i}/2)(\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\sigma_{\rho\epsilon}\Psi) - m^*S_{0\rho\epsilon} \tag{B.46}$$

$$(g/2)[\epsilon_{\rho\epsilon}{}^{\nu\sigma}J^{b}{}_{\sigma} + \mathrm{i}(\delta_{\epsilon}{}^{\nu}\delta_{\rho}{}^{\sigma} - \delta_{\rho}{}^{\nu}\delta_{\epsilon}{}^{\sigma})K^{b}{}_{\sigma}]W_{b\nu} = \mathrm{i}\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\partial\!\!\!/\Psi - m^*S_{0\rho\epsilon} \tag{B.47}$$

Multiply by $\overline{\Psi}\tau_a\gamma_5\sigma_{\rho\epsilon}$:

$$\begin{aligned}(g/2)[\mathrm{i}(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})K_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}\epsilon_{\rho\epsilon}{}^{\nu\sigma}\epsilon_{a}{}^{bc}J_{c\sigma}]W_{b\nu}\\ = (\mathrm{i}/2)(\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi - \overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\sigma_{\rho\epsilon}\tau_a\Psi) - m^*S_{a\rho\epsilon}\end{aligned} \tag{B.48}$$

$$\overline{\Psi}\overleftarrow{\partial\!\!\!/}\gamma_5\sigma_{\rho\epsilon}\tau_a\Psi = g[(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})\mathrm{i}\epsilon_{a}{}^{bc}K_{c\sigma} - \mathrm{i}\epsilon_{\rho\epsilon}{}^{\nu\sigma}J_{0\sigma}\delta_{a}{}^{b}]W_{b\nu} - \overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi \tag{B.49}$$

$$\begin{aligned}(g/2)[\epsilon_{\rho\epsilon}{}^{\nu\sigma}(J_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}J_{c\sigma}\epsilon_{a}{}^{bc}) + \mathrm{i}(\delta_{\rho}{}^{\sigma}\delta_{\epsilon}{}^{\nu} - \delta_{\epsilon}{}^{\sigma}\delta_{\rho}{}^{\nu})(K_{0\sigma}\delta_{a}{}^{b} + \mathrm{i}K_{c\sigma}\epsilon_{a}{}^{bc})]W_{b\nu}\\ = \mathrm{i}\overline{\Psi}\gamma_5\sigma_{\rho\epsilon}\tau_a\partial\!\!\!/\Psi - m^*S_{a\rho\epsilon}\end{aligned} \tag{B.50}$$

Appendix C

Vector potential in tensor form

This appendix contains a more detailed derivation of the inverted Dirac equation in terms of bilinear tensors only, to supplement the brief outline contained in section 2.3. Throughout, we will make heavy use of the identities contained within appendix A.

Given the two different forms of the inverted Abelian Dirac equation

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] + \partial_\nu s_\mu{}^\nu - 2mj_\mu}{\sigma} \tag{C.1}$$

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] + \partial_\nu {}^*s_\mu{}^\nu}{\omega}, \tag{C.2}$$

we can combine these into a single equation by adding them together and dividing by 2

$$A_\mu = \frac{1}{4q}\left\{\frac{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega + \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma}{\sigma\omega} + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega}\right.$$
$$\left. - \frac{2mj_\mu}{\sigma}\right\}. \tag{C.3}$$

The appropriate tensor forms needed to replace the spinor terms in (C.3) are $j^\nu(\partial_\mu k_\nu)$ and $m^\nu(\partial_\mu n_\nu)$. Consider the first

$$j^\nu(\partial_\mu k_\nu) = \overline{\psi}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi + \overline{\psi}\gamma^\nu\psi \cdot \overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi). \tag{C.4}$$

Fierz expanding the first term gives

$$\overline{\psi}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi = -(\partial_\mu\overline{\psi})\psi \cdot \omega - (1/2)(\partial_\mu\overline{\psi})\gamma_\sigma\psi \cdot k^\sigma$$
$$- (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi \cdot j^\sigma + (\partial_\mu\overline{\psi})\gamma_5\psi \cdot \sigma, \tag{C.5}$$

and the second term gives

$$\overline{\psi}\gamma^\nu\psi \cdot \overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) = -\overline{\psi}\gamma_5(\partial_\mu\psi) \cdot \sigma - (1/2)\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi) \cdot j^\sigma$$
$$- (1/2)\overline{\psi}\gamma_\sigma(\partial_\mu\psi) \cdot k^\sigma + \overline{\psi}(\partial_\mu\psi) \cdot \omega. \tag{C.6}$$

Combining these, and rearranging the equation gives

$$j^\nu(\partial_\mu k_\nu) = (2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma - (1/3)k^\nu(\partial_\mu j_\nu). \tag{C.7}$$

We can see that in order to completely derive this Fierz identity, we need to calculate $k^\nu(\partial_\mu j_\nu)$, and eliminate it by substitution. This is the same situation that arises when calculating the Fierz identities for $j^\nu j_\nu$ and $k^\nu k_\nu$.

$$k^\nu(\partial_\mu j_\nu) = \overline{\psi}\gamma_5\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_\nu\psi + \overline{\psi}\gamma_5\gamma^\nu\psi \cdot \overline{\psi}\gamma_\nu(\partial_\mu\psi), \tag{C.8}$$

now Fierz expanding the two terms respectively

$$\begin{aligned}\overline{\psi}\gamma_5\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_\nu\psi &= (\partial_\mu\overline{\psi})\psi \cdot \omega - (1/2)(\partial_\mu\overline{\psi})\gamma_\sigma\psi \cdot k^\sigma \\ &\quad - (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi \cdot j^\sigma - (\partial_\mu\overline{\psi})\gamma_5\psi \cdot \sigma, \qquad (C.9)\\ \overline{\psi}\gamma_5\gamma^\nu\psi \cdot \overline{\psi}\gamma_\nu(\partial_\mu\psi) &= -\overline{\psi}(\partial_\mu\psi) \cdot \omega - (1/2)\overline{\psi}\gamma_\sigma(\partial_\mu\psi) \cdot k^\sigma \\ &\quad - (1/2)\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi) \cdot j^\sigma + \overline{\psi}\gamma_5(\partial_\mu\psi) \cdot \sigma. \qquad (C.10)\end{aligned}$$

Adding the terms gives

$$\begin{aligned}k^\nu(\partial_\mu j_\nu) &= (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma - (2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega \\ &\quad - (1/3)j^\nu(\partial_\mu k_\nu), \qquad (C.11)\end{aligned}$$

and substituting this into the $j^\nu(\partial_\mu k_\nu)$ identity gives

$$j^\nu(\partial_\mu k_\nu) = [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma. \tag{C.12}$$

Likewise, substituting this into the $k^\nu(\partial_\mu j_\nu)$ identity gives

$$k^\nu(\partial_\mu j_\nu) = [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma - [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega, \tag{C.13}$$

which implies the new Fierz identity

$$j^\nu(\partial_\mu k_\nu) = -k^\nu(\partial_\mu j_\nu) = [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma. \tag{C.14}$$

Now we consider $m^\nu(\partial_\mu n_\nu)$, which, after applying the derivative and expanding is

$$\begin{aligned}m^\nu(\partial_\mu n_\nu) &= (\mathrm{i}/4)[\overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_\nu\psi^{\mathrm{c}} + \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot \overline{\psi}\gamma_\nu(\partial_\mu\psi^{\mathrm{c}}) \\ &\quad - \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\nu\psi - \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot \overline{\psi}^{\mathrm{c}}\gamma_\nu(\partial_\mu\psi) + \overline{\psi}\gamma^\nu\psi^{\mathrm{c}} \cdot (\partial_\mu\overline{\psi})\gamma_\nu\psi^{\mathrm{c}} \\ &\quad + \overline{\psi}\gamma^\nu\psi^{\mathrm{c}} \cdot \overline{\psi}\gamma_\nu(\partial_\mu\psi^{\mathrm{c}}) - \overline{\psi}\gamma^\nu\psi^{\mathrm{c}} \cdot (\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\nu\psi - \overline{\psi}\gamma^\nu\psi^{\mathrm{c}} \cdot \overline{\psi}^{\mathrm{c}}\gamma_\nu(\partial_\mu\psi)]. \qquad (C.15)\end{aligned}$$

After Fierz expanding and applying the charge conjugation identity (A.35), the eight terms respectively are

$$\begin{aligned}\overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi})\gamma_\nu\psi^{\mathrm{c}} &= -(\partial_\mu\overline{\psi})\psi \cdot \sigma - (1/2)(\partial_\mu\overline{\psi})\gamma_\sigma\psi \cdot j^\sigma \\ &\quad + (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi \cdot k^\sigma + (\partial_\mu\overline{\psi})\gamma_5\psi \cdot \omega, \qquad (C.16)\\ \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot \overline{\psi}\gamma_\nu(\partial_\mu\psi^{\mathrm{c}}) &= -(\partial_\mu\overline{\psi})\psi \cdot \sigma - (1/2)(\partial_\mu\overline{\psi})\gamma_\sigma\psi \cdot j^\sigma \\ &\quad + (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi \cdot k^\sigma + (\partial_\mu\overline{\psi})\gamma_5\psi \cdot \omega, \qquad (C.17)\\ \overline{\psi}^{\mathrm{c}}\gamma^\nu\psi \cdot (\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\nu\psi &= (\partial_\mu\overline{\psi}^{\mathrm{c}})\psi \cdot \overline{\psi}^{\mathrm{c}}\psi - (1/2)(\partial_\mu\overline{\psi}^{\mathrm{c}})\gamma_\sigma\psi \cdot \overline{\psi}^{\mathrm{c}}\gamma^\sigma\psi\end{aligned}$$

$$- (1/2)(\partial_\mu\overline{\psi}^{\rm c})\gamma_5\gamma_\sigma\psi\cdot\overline{\psi}^{\rm c}\gamma_5\gamma^\sigma\psi - (\partial_\mu\overline{\psi}^{\rm c})\gamma_5\psi\cdot\overline{\psi}^{\rm c}\gamma_5\psi, \tag{C.18}$$

$$\overline{\psi}^{\rm c}\gamma^\nu\psi\cdot\overline{\psi}^{\rm c}\gamma_\nu(\partial_\mu\psi) = \overline{\psi}^{\rm c}\psi\cdot\overline{\psi}^{\rm c}(\partial_\mu\psi) - (1/2)\overline{\psi}^{\rm c}\gamma_\sigma\psi\cdot\overline{\psi}^{\rm c}\gamma^\sigma(\partial_\mu\psi)$$
$$- (1/2)\overline{\psi}^{\rm c}\gamma_5\gamma_\sigma\psi\cdot\overline{\psi}^{\rm c}\gamma_5\gamma^\sigma(\partial_\mu\psi) - \overline{\psi}^{\rm c}\gamma_5\psi\cdot\overline{\psi}^{\rm c}\gamma_5(\partial_\mu\psi), \tag{C.19}$$

$$\overline{\psi}\gamma^\nu\psi^{\rm c}\cdot(\partial_\mu\overline{\psi})\gamma_\nu\psi^{\rm c} = (\partial_\mu\overline{\psi})\psi^{\rm c}\cdot\overline{\psi}\psi^{\rm c} - (1/2)(\partial_\mu\overline{\psi})\gamma_\sigma\psi^{\rm c}\cdot\overline{\psi}\gamma^\sigma\psi^{\rm c}$$
$$- (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi^{\rm c}\cdot\overline{\psi}\gamma_5\gamma^\sigma\psi^{\rm c} - (\partial_\mu\overline{\psi})\gamma_5\psi^{\rm c}\cdot\overline{\psi}\gamma_5\psi^{\rm c}, \tag{C.20}$$

$$\overline{\psi}\gamma^\nu\psi^{\rm c}\cdot\overline{\psi}\gamma_\nu(\partial_\mu\psi^{\rm c}) = \overline{\psi}\psi^{\rm c}\cdot\overline{\psi}(\partial_\mu\psi^{\rm c}) - (1/2)\overline{\psi}\gamma_\sigma\psi^{\rm c}\cdot\overline{\psi}\gamma^\sigma(\partial_\mu\psi^{\rm c})$$
$$- (1/2)\overline{\psi}\gamma_5\gamma_\sigma\psi^{\rm c}\cdot\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi^{\rm c}) - \overline{\psi}\gamma_5\psi^{\rm c}\cdot\overline{\psi}\gamma_5(\partial_\mu\psi^{\rm c}), \tag{C.21}$$

$$\overline{\psi}\gamma^\nu\psi^{\rm c}\cdot(\partial_\mu\overline{\psi}^{\rm c})\gamma_\nu\psi = -\overline{\psi}(\partial_\mu\psi)\sigma - (1/2)\overline{\psi}\gamma_\sigma(\partial_\mu\psi)\cdot j^\sigma$$
$$+ (1/2)\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi)\cdot k^\sigma + \overline{\psi}\gamma_5(\partial_\mu\psi)\cdot\omega, \tag{C.22}$$

$$\overline{\psi}\gamma^\nu\psi^{\rm c}\cdot\overline{\psi}^{\rm c}\gamma_\nu(\partial_\mu\psi) = -\overline{\psi}(\partial_\mu\psi)\cdot\sigma - (1/2)\overline{\psi}\gamma_\sigma(\partial_\mu\psi)\cdot j^\sigma$$
$$+ (1/2)\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi)\cdot k^\sigma + \overline{\psi}\gamma_5(\partial_\mu\psi)\cdot\omega. \tag{C.23}$$

Adding these together and gathering terms gives

$$\begin{aligned} m^\nu(\partial_\mu n_\nu) = (\mathrm{i}/4)\{&2[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - 2[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega \\ &+ [\overline{\psi}\gamma_\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\sigma\psi]j^\sigma - [\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi]k^\sigma \\ &- \partial_\mu(\overline{\psi}^{\rm c}\psi)\cdot\overline{\psi}^{\rm c}\psi + \partial_\mu(\overline{\psi}\psi^{\rm c})\cdot\overline{\psi}\psi^{\rm c} + \partial_\mu(\overline{\psi}^{\rm c}\gamma_5\psi)\cdot\overline{\psi}^{\rm c}\gamma_5\psi \\ &- \partial_\mu(\overline{\psi}\gamma_5\psi^{\rm c})\cdot\overline{\psi}\gamma_5\psi^{\rm c} + (1/2)\partial_\mu(\overline{\psi}^{\rm c}\gamma_\sigma\psi)\cdot\overline{\psi}^{\rm c}\gamma^\sigma\psi \\ &- (1/2)\partial_\mu(\overline{\psi}\gamma_\sigma\psi^{\rm c})\cdot\overline{\psi}\gamma^\sigma\psi^{\rm c} + (1/2)\partial_\mu(\overline{\psi}^{\rm c}\gamma_5\gamma_\sigma\psi)\cdot\overline{\psi}^{\rm c}\gamma_5\gamma^\sigma\psi \\ &- (1/2)\partial_\mu(\overline{\psi}\gamma_5\gamma_\sigma\psi^{\rm c})\cdot\overline{\psi}\gamma_5\gamma^\sigma\psi^{\rm c}\}. \end{aligned} \tag{C.24}$$

Now, using (A.35), and setting $\chi^{\rm c} = \psi$, which implies $\overline{\chi} = \overline{\psi}^{\rm c}$, we can show that $\overline{\psi}^{\rm c}\psi = 0$, $\overline{\psi}^{\rm c}\gamma_5\gamma_\sigma\psi = 0$ and $\overline{\psi}^{\rm c}\gamma_5\psi = 0$, because they equal their own negatives. The same goes for the corresponding bilinears with the charge conjugation index switched. Our equation now whittles down to

$$\begin{aligned} m^\nu(\partial_\mu n_\nu) = (\mathrm{i}/4)\{&2[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - 2[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega \\ &+ [\overline{\psi}\gamma_\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\sigma\psi]j^\sigma - [\overline{\psi}\gamma_5\gamma_\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\sigma\psi]k^\sigma \\ &+ (1/2)\partial_\mu(\overline{\psi}^{\rm c}\gamma_\sigma\psi)\cdot\overline{\psi}^{\rm c}\gamma^\sigma\psi - (1/2)\partial_\mu(\overline{\psi}\gamma_\sigma\psi^{\rm c})\cdot\overline{\psi}\gamma^\sigma\psi^{\rm c}\}. \end{aligned} \tag{C.25}$$

We can use the result of the complex conjugation bilinear identity (A.50), $(\overline{\psi}^{\rm c}\gamma_\mu\psi)^* = \overline{\psi}\gamma_\mu\psi^{\rm c}$, which implies that $\overline{\psi}\gamma_\mu\psi^{\rm c} = m_\mu - \mathrm{i}n_\mu$. So taking the last two terms from the equation for $m^\nu(\partial_\mu n_\nu)$

$$\begin{aligned} &(1/2)\partial_\mu(\overline{\psi}^{\rm c}\gamma_\sigma\psi)\cdot\overline{\psi}^{\rm c}\gamma^\sigma\psi - (1/2)\partial_\mu(\overline{\psi}\gamma_\sigma\psi^{\rm c})\cdot\overline{\psi}\gamma^\sigma\psi^{\rm c} \\ &\quad = (1/2)\partial_\mu(m_\sigma + \mathrm{i}n_\sigma)(m^\sigma + \mathrm{i}n^\sigma) - (1/2)\partial_\mu(m_\sigma - \mathrm{i}n_\sigma)(m^\sigma - \mathrm{i}n^\sigma) \\ &\quad = 2\mathrm{i}\partial_\mu(m_\sigma n^\sigma) \\ &\quad = 0, \end{aligned} \tag{C.26}$$

by the orthogonality of m_μ and n_ν. Our equation now becomes

$$m^\nu(\partial_\mu n_\nu) = (\mathrm{i}/4)\{2[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - 2[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega$$

$$+ [\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu - [\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu\}. \tag{C.27}$$

We can eliminate the last two terms via a Fierz expansion process analogous to that of $j^\nu(\partial_\mu k_\nu)$ and $k^\nu(\partial_\mu j_\nu)$. The expanded terms on the left are

$$\begin{aligned}\overline{\psi}\gamma_\nu(\partial_\mu\psi)\cdot\overline{\psi}\gamma^\nu\psi &= \overline{\psi}(\partial_\mu\psi)\cdot\sigma - (1/2)\overline{\psi}\gamma_\nu(\partial_\mu\psi)\cdot j^\nu - (1/2)\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi)\cdot k^\nu \\ &\quad - \overline{\psi}\gamma_5(\partial_\mu\psi)\cdot\omega, \qquad \text{(C.28)}\\ (\partial_\mu\overline{\psi})\gamma_\nu\psi\cdot\overline{\psi}\gamma^\nu\psi &= (\partial_\mu\overline{\psi})\psi\cdot\sigma - (1/2)(\partial_\mu\overline{\psi})\gamma_\nu\psi\cdot j^\nu - (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi\cdot k^\sigma \\ &\quad - (\partial_\mu\overline{\psi})\gamma_5\psi\cdot\omega. \qquad \text{(C.29)}\end{aligned}$$

Subtracting the second term from the first and rearranging gives

$$\begin{aligned}&[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu = (2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma \\ &\quad - (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega - (1/3)[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu,\end{aligned} \tag{C.30}$$

where we can see that as mentioned before, we must also consider the Fierz expansion of $[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu$. The two Fierz expanded terms are

$$\begin{aligned}\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi)\cdot\overline{\psi}\gamma_5\gamma^\nu\psi &= -\overline{\psi}(\partial_\mu\psi)\cdot\sigma - (1/2)\overline{\psi}\gamma_\nu(\partial_\mu\psi)\cdot j^\nu \\ &\quad - (1/2)\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi)\cdot k^\nu + \overline{\psi}\gamma_5(\partial_\mu\psi)\cdot\omega, \qquad \text{(C.31)}\\ (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi\cdot\overline{\psi}\gamma_5\gamma^\nu\psi &= -(\partial_\mu\overline{\psi})\psi\cdot\sigma - (1/2)(\partial_\mu\overline{\psi})\gamma_\nu\psi\cdot j^\nu \\ &\quad - (1/2)(\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi\cdot k^\nu + (\partial_\mu\overline{\psi})\gamma_5\psi\cdot\omega. \qquad \text{(C.32)}\end{aligned}$$

Subtracting the second term from the first and rearranging gives

$$\begin{aligned}&[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu = -(2/3)[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma \\ &\quad + (2/3)[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega - (1/3)[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu.\end{aligned} \tag{C.33}$$

Substituting this into (C.30) gives the identity

$$[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu = [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega, \tag{C.34}$$

then subsequently substituting this into (C.33) gives

$$[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu = -[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma + [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega. \tag{C.35}$$

This provides us with another Fierz identity

$$\begin{aligned}[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]j^\nu &= -[\overline{\psi}\gamma_5\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma_\nu\psi]k^\nu \\ &= [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega,\end{aligned} \tag{C.36}$$

which when substituted into (C.27) gives its final form

$$m^\nu(\partial_\mu n_\nu) = \mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma - \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\omega. \tag{C.37}$$

It can be shown via a similar process, that

$$m^\nu(\partial_\mu n_\nu) = -n^\nu(\partial_\mu m_\nu). \tag{C.38}$$

We can now use the identities (C.12) and (C.37) to describe the spinorial objects in the inverted Dirac equation in terms of tensors alone. From (C.12), we get

$$[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] = j^\nu(\partial_\mu k_\nu)\omega^{-1} + [\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma\omega^{-1}, \tag{C.39}$$

and from (C.37) we get

$$[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] = \mathrm{i}m^\nu(\partial_\mu n_\nu)\omega^{-1} + [\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\sigma\omega^{-1}. \tag{C.40}$$

After substitution and rearrangement, we get the required spinor replacement identities

$$[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] = -(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\omega + \mathrm{i}m^\nu(\partial_\mu n_\nu)\sigma], \tag{C.41}$$
$$[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] = -(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\sigma + \mathrm{i}m^\nu(\partial_\mu n_\nu)\omega]. \tag{C.42}$$

Combining these two identities in the form they appear in (C.3)

$$\begin{aligned}
&\{\mathrm{i}[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi]\omega + \mathrm{i}[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi]\sigma\}(\sigma\omega)^{-1} \\
&\quad = \{\omega[m^\nu(\partial_\mu n_\nu)\sigma - \mathrm{i}j^\nu(\partial_\mu k_\nu)\omega] + \sigma[m^\nu(\partial_\mu n_\nu)\omega - \mathrm{i}j^\nu(\partial_\mu k_\nu)\sigma]\} \\
&\qquad \cdot [\sigma\omega(\sigma^2-\omega^2)]^{-1} \\
&\quad = \frac{2m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \frac{\mathrm{i}j^\nu(\partial_\mu k_\nu)}{\sigma^2-\omega^2}\left(\frac{\sigma^2+\omega^2}{\sigma\omega}\right).
\end{aligned} \tag{C.43}$$

Substitute into (C.3):

$$A_\mu = \frac{1}{4q}\left\{\frac{2m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\sigma^2+\omega^2}{\sigma\omega(\sigma^2-\omega^2)}\right] + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega} - \frac{2mj_\mu}{\sigma}\right\}, \tag{C.44}$$

which technically, is an expression for A_μ exclusively in tensor form. However, we can simplify this. Substituting (C.41) into the original inverted Dirac equation (C.1) gives

$$A_\mu = \frac{1}{2q}\left\{\frac{m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\omega}{\sigma(\sigma^2-\omega^2)}\right] + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} - \frac{2mj_\mu}{\sigma}\right\}, \tag{C.45}$$

and substituting (C.42) into the alternative inverted Dirac equation (C.2) gives

$$A_\mu = \frac{1}{2q}\left\{\frac{m^\nu(\partial_\mu n_\nu)}{\sigma^2-\omega^2} - \mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\sigma}{\omega(\sigma^2-\omega^2)}\right] + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega}\right\}. \tag{C.46}$$

Adding (C.45) and (C.46) and dividing by 2 results in (C.44), the combined form of A_μ already obtained. But if we *subtract* these two and rearrange, we obtain a new identity that we can use to eliminate $\mathrm{i}j^\nu(\partial_\mu k_\nu)$

$$\mathrm{i}j^\nu(\partial_\mu k_\nu) = 2m\omega j_\mu + \sigma\partial_\nu {}^*s_\mu{}^\nu - \omega\partial_\nu s_\mu{}^\nu. \tag{C.47}$$

Take the $\mathrm{i}j^\nu(\partial_\mu k_\nu)$ term from (C.44) and substitute the above identity

$$-\mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\sigma^2+\omega^2}{\sigma\omega(\sigma^2-\omega^2)}\right] = -(2m\omega j_\mu + \sigma\partial_\nu {}^*s_\mu{}^\nu - \omega\partial_\nu s_\mu{}^\nu)\left[\frac{\sigma^2+\omega^2}{\sigma\omega(\sigma^2-\omega^2)}\right]$$

$$= \frac{-2m\sigma^2\omega j_\mu - 2m\omega^3 j_\mu - \sigma^3\partial_\nu {}^*s_\mu{}^\nu - \sigma\omega^2\partial_\nu {}^*s_\mu{}^\nu + \sigma^2\omega\partial_\nu s_\mu{}^\nu + \omega^3\partial_\nu s_\mu{}^\nu}{\sigma\omega(\sigma^2-\omega^2)}. \tag{C.48}$$

Rearrange the last three terms in (C.44)

$$\frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega} - \frac{2mj_\mu}{\sigma} = \left(\frac{\omega\partial_\nu s_\mu{}^\nu + \sigma\partial_\nu {}^*s_\mu{}^\nu - 2m\omega j_\mu}{\sigma\omega}\right)\left(\frac{\sigma^2-\omega^2}{\sigma^2-\omega^2}\right)$$
$$= \frac{-2m\sigma^2\omega j_\mu + 2m\omega^3 j_\mu + \sigma^3\partial_\nu {}^*s_\mu{}^\nu - \sigma\omega^2\partial_\nu {}^*s_\mu{}^\nu + \sigma^2\omega\partial_\nu s_\mu{}^\nu - \omega^3\partial_\nu s_\mu{}^\nu}{\sigma\omega(\sigma^2-\omega^2)}, \tag{C.49}$$

and adding the previous two equations together gives

$$\begin{aligned} &-\mathrm{i}j^\nu(\partial_\mu k_\nu)\left[\frac{\sigma^2+\omega^2}{\sigma\omega(\sigma^2-\omega^2)}\right] + \frac{\partial_\nu s_\mu{}^\nu}{\sigma} + \frac{\partial_\nu {}^*s_\mu{}^\nu}{\omega} - \frac{2mj_\mu}{\sigma} \\ &= \frac{2\sigma^2\omega\partial_\nu s_\mu{}^\nu - 2\sigma\omega^2\partial_\nu {}^*s_\mu{}^\nu - 4m\sigma^2\omega j_\mu}{\sigma\omega(\sigma^2-\omega^2)} \\ &= \frac{2(\sigma\partial_\nu s_\mu{}^\nu - \omega\partial_\nu {}^*s_\mu{}^\nu - 2m\sigma j_\mu)}{(\sigma^2-\omega^2)}. \end{aligned} \tag{C.50}$$

Therefore, our final form of A_μ in tensor form is

$$A_\mu = \frac{1}{2q}\frac{m^\nu(\partial_\mu n_\nu) + \sigma\partial_\nu s_\mu{}^\nu - \omega\partial_\nu {}^*s_\mu{}^\nu - 2m\sigma j_\mu}{\sigma^2-\omega^2}. \tag{C.51}$$

Appendix D

Cylindrically symmetric field strength tensor

In this appendix, we present the dependent functions of t and ρ in the cylindrically symmetric field strength tensor in terms of the j^μ, k^μ, σ and ω tensor fields. The functions appearing in $F_{01} = -xF_a + yF_b$ and $F_{02} = -yF_a - xF_b$ are

$$
\begin{aligned}
F_a = {} & -2[q(\sigma^2-\omega^2)^3]^{-1}\{(j_d k_{c,t} - j_c k_{d,t} - k_d j_{c,t} + k_c j_{d,t} - 2m\sigma j_b)(\sigma\sigma_t - \omega\omega_t) \\
& \cdot (\sigma^2-\omega^2) + (j_c k_{d,\rho} - j_d k_{c,\rho} - k_c j_{d,\rho} + k_d j_{c,\rho})(\sigma\sigma_\rho - \omega\omega_\rho)(\sigma^2-\omega^2) \\
& + (j_c k_d - j_d k_c)(\sigma\sigma_t - \omega\omega_t)^2 + (j_d k_c - j_c k_d)(\sigma\sigma_\rho - \omega\omega_\rho)^2 + (2/\rho)(j_c k_d \\
& - j_d k_c - m\sigma j_a)(\sigma\sigma_\rho - \omega\omega_\rho)(\sigma^2-\omega^2) + \mathrm{i}[(j_b k_a - j_a k_b)(\sigma_t\omega - \sigma\omega_t) \\
& \cdot (\sigma\sigma_t - \omega\omega_t) + (j_a k_b - j_b k_a)(\sigma_\rho\omega - \sigma\omega_\rho)(\sigma\sigma_\rho - \omega\omega_\rho)]\} \\
& + [2q(\sigma^2-\omega^2)^2]^{-1}\{(j_d k_{c,tt} + 2j_{d,t}k_{c,t} + j_{d,tt}k_c - j_c k_{d,tt} - 2j_{c,t}k_{d,t} \\
& - j_{c,tt}k_d + j_c k_{d,\rho\rho} + 2j_{c,\rho}k_{d,\rho} + j_{c,\rho\rho}k_d - j_d k_{c,\rho\rho} - 2j_{d,\rho}k_{c,\rho} - j_{d,\rho\rho}k_c \\
& - 2m\sigma_t j_b - 2m\sigma j_{b,t})(\sigma^2-\omega^2) + (j_d k_{c,t} + j_{d,t}k_c - j_c k_{d,t} - j_{c,t}k_d \\
& - 4m\sigma j_b)(\sigma\sigma_t - \omega\omega_t) + (j_c k_{d,\rho} + j_{c,\rho}k_d - j_d k_{c,\rho} - j_{d,\rho}k_c)(\sigma\sigma_\rho - \omega\omega_\rho) \\
& + (j_c k_d - j_d k_c)(\sigma_t^2 + \sigma\sigma_{tt} - \omega_t^2 - \omega\omega_{tt}) + (j_d k_c - j_c k_d)(\sigma_\rho^2 + \sigma\sigma_{\rho\rho} \\
& - \omega_\rho^2 - \omega\omega_{\rho\rho}) + (1/\rho)(3j_c k_{d,\rho} + 3j_{c,\rho}k_d - 3j_d k_{c,\rho} - 3j_{d,\rho}k_c - 2m\sigma_\rho j_a \\
& - 2m\sigma j_{a,\rho})(\sigma^2-\omega^2) + (1/\rho)(3j_c k_d - 3j_d k_c - 4m\sigma j_a)(\sigma\sigma_\rho - \omega\omega_\rho) \\
& + \mathrm{i}[(j_b k_{a,t} + j_{b,t}k_a - j_a k_{b,t} - j_{a,t}k_b)(\sigma_t\omega - \sigma\omega_t) + (j_a k_{b,\rho} + j_{a,\rho}k_b \\
& - j_b k_{a,\rho} - j_{b,\rho}k_a)(\sigma_\rho\omega - \sigma\omega_\rho) + (j_b k_a - j_a k_b)(\sigma_{tt}\omega - \sigma\omega_{tt}) + (j_a k_b \\
& - j_b k_a)(\sigma_{\rho\rho}\omega - \sigma\omega_{\rho\rho}) + (1/\rho)(j_a k_b - j_b k_a)(\sigma_\rho\omega - \sigma\omega_\rho)] + (j_d k_a \\
& - j_a k_d)(j_b j_{c,t} - j_c j_{b,t} - k_b k_{c,t} + k_c k_{b,t}) + (j_c k_b - j_b k_c)(j_d j_{a,t} - j_a j_{d,t} \\
& - k_d k_{a,t} + k_a k_{d,t}) - \rho[(j_a k_b - j_b k_a)(j_{d,t}j_{c,\rho} - j_{c,t}j_{d,\rho} - k_{d,t}k_{c,\rho} + k_{c,t}k_{d,\rho}) \\
& + (j_a k_c - j_c k_a)(j_{b,t}j_{d,\rho} - j_{d,t}j_{b,\rho} - k_{b,t}k_{d,\rho} + k_{d,t}k_{b,\rho}) + (j_a k_d - j_d k_a) \\
& \cdot (j_{c,t}j_{b,\rho} - j_{b,t}j_{c,\rho} - k_{c,t}k_{b,\rho} + k_{b,t}k_{c,\rho}) + (j_b k_c - j_c k_b)(j_{d,t}j_{a,\rho} - j_{a,t}j_{d,\rho} \\
& - k_{d,t}k_{a,\rho} + k_{a,t}k_{d,\rho}) + (j_b k_d - j_d k_b)(j_{a,t}j_{c,\rho} - j_{c,t}j_{a,\rho} - k_{a,t}k_{c,\rho} + k_{c,t}k_{a,\rho})
\end{aligned}
$$

$$+ (j_c k_d - j_d k_c)(j_{b,t} j_{a,\rho} - j_{a,t} j_{b,\rho} - k_{b,t} k_{a,\rho} + k_{a,t} k_{b,\rho})]\}, \tag{D.1}$$

as well as

$$\begin{aligned}
F_b = &-2[q(\sigma^2-\omega^2)^3]^{-1}(\sigma\sigma_t - \omega\omega_t)\{(j_b k_{d,t} + j_{b,t} k_d - j_d k_{b,t} - j_{d,t} k_b - 2m\sigma j_c) \\
&\cdot (\sigma^2-\omega^2) + (j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) + (1/\rho)(j_a k_{d,\rho} + j_{a,\rho} k_d - j_d k_{a,\rho} \\
&- j_{d,\rho} k_a)(\sigma^2-\omega^2) + (1/\rho)(j_d k_a - j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \mathrm{i}[(j_c k_a - j_a k_c) \\
&\cdot (\sigma_t\omega - \sigma\omega_t) + \rho(j_c k_b - j_b k_c)(\sigma_\rho\omega - \sigma\omega_\rho)]\} + [2q(\sigma^2-\omega^2)^2]^{-1}\{(j_b k_{d,tt} \\
&+ 2j_{b,t} k_{d,t} + j_{b,tt} k_d - j_d k_{b,tt} - 2j_{d,t} k_{b,t} - j_{d,tt} k_b - 2m\sigma_t j_c - 2m\sigma j_{c,t}) \\
&\cdot (\sigma^2-\omega^2) + (j_b k_{d,t} + j_{b,t} k_d - j_d k_{b,t} - j_{d,t} k_b - 4m\sigma j_c)(\sigma\sigma_t - \omega\omega_t) \\
&+ (j_d k_b - j_b k_d)(\sigma_t^2 + \sigma\sigma_{tt} - \omega_t^2 - \omega\omega_{tt}) + (1/\rho)(j_a k_{d,t\rho} + j_{a,t} k_{d,\rho} \\
&+ j_{a,\rho} k_{d,t} + j_{a,t\rho} k_d - j_d k_{a,t\rho} - j_{d,t} k_{a,\rho} - j_{d,\rho} k_{a,t} - j_{d,t\rho} k_a)(\sigma^2-\omega^2) \\
&+ (2/\rho)(j_a k_{d,\rho} + j_{a,\rho} k_d - j_d k_{a,\rho} - j_{d,\rho} k_a)(\sigma\sigma_t - \omega\omega_t) + (1/\rho)(j_d k_{a,t} \\
&+ j_{d,t} k_a - j_a k_{d,t} - j_{a,t} k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + (1/\rho)(j_d k_a - j_a k_d)(\sigma_t\sigma_\rho + \sigma\sigma_{t\rho} \\
&- \omega_t\omega_\rho - \omega\omega_{t\rho}) + \mathrm{i}[(j_c k_{a,t} + j_{c,t} k_a - j_a k_{c,t} - j_{a,t} k_c)(\sigma_t\omega - \sigma\omega_t) \\
&+ (j_c k_a - j_a k_c)(\sigma_{tt}\omega - \sigma\omega_{tt}) + \rho(j_c k_{b,t} + j_{c,t} k_b - j_b k_{c,t} - j_{b,t} k_c)(\sigma_\rho\omega \\
&- \sigma\omega_\rho) + \rho(j_c k_b - j_b k_c)(\sigma_\rho\omega_t + \sigma_{t\rho}\omega - \sigma_t\omega_\rho - \sigma\omega_{t\rho}) + (j_a k_d - j_d k_a) \\
&\cdot (j_b j_{b,t} + j_c j_{c,t} - k_b k_{b,t} - k_c k_{c,t}) + (j_b k_b + j_c k_c)(j_{a,t} j_d - j_a j_{d,t} + k_{a,t} k_d \\
&- k_a k_{d,t}) + (j_b^2 + j_c^2)(j_{d,t} k_a - j_{a,t} k_d) + (k_b^2 + k_c^2)(j_a k_{d,t} - j_d k_{a,t})]\}.
\end{aligned} \tag{D.2}$$

The function in $F_{03} = -F_c$ is

$$\begin{aligned}
F_c = &\, 2[q(\sigma^2-\omega^2)^3]^{-1}(\sigma\sigma_t - \omega\omega_t)\{2(j_c k_a - j_a k_c - m\sigma j_d)(\sigma^2-\omega^2) \\
&+ \rho(j_c k_{a,\rho} + j_{c,\rho} k_a - j_a k_{c,\rho} - j_{a,\rho} k_c)(\sigma^2-\omega^2) + \rho(j_a k_c - j_c k_a)(\sigma\sigma_\rho \\
&- \omega\omega_\rho) + \rho^2(j_c k_{b,t} + j_{c,t} k_b - j_b k_{c,t} - j_{b,t} k_c)(\sigma^2-\omega^2) + \rho^2(j_b k_c - j_c k_b) \\
&\cdot (\sigma\sigma_t - \omega\omega_t) + \mathrm{i}[(j_d k_a - j_a k_d)(\sigma_t\omega - \sigma\omega_t) + \rho(j_d k_b - j_b k_d)(\sigma_\rho\omega - \sigma\omega_\rho)]\} \\
&- [2q(\sigma^2-\omega^2)^2]^{-1}\{2(j_c k_{a,t} + j_{c,t} k_a - j_a k_{c,t} - j_{a,t} k_c - m\sigma_t j_d - m\sigma j_{d,t}) \\
&\cdot (\sigma^2-\omega^2) + 4(j_c k_a - j_a k_c - m\sigma j_d)(\sigma\sigma_t - \omega\omega_t) + \rho(j_c k_{a,t\rho} + j_{c,t} k_{a,\rho} \\
&+ j_{c,\rho} k_{a,t} + j_{c,t\rho} k_a - j_a k_{c,t\rho} - j_{a,t} k_{c,\rho} - j_{a,\rho} k_{c,t} - j_{a,t\rho} k_c)(\sigma^2-\omega^2) \\
&+ 2\rho(j_c k_{a,\rho} + j_{c,\rho} k_a - j_a k_{c,\rho} - j_{a,\rho} k_c)(\sigma\sigma_t - \omega\omega_t) + \rho(j_a k_{c,t} + j_{a,t} k_c \\
&- j_c k_{a,t} - j_{c,t} k_a)(\sigma\sigma_\rho - \omega\omega_\rho) + \rho(j_a k_c - j_c k_a)(\sigma_t\sigma_\rho + \sigma\sigma_{t\rho} - \omega_t\omega_\rho \\
&- \omega\omega_{t\rho}) + \rho^2(j_c k_{b,tt} + 2j_{c,t} k_{b,t} + j_{c,tt} k_b - j_b k_{c,tt} - 2j_{b,t} k_{c,t} - j_{b,tt} k_c) \\
&\cdot (\sigma^2-\omega^2) + \rho^2(j_c k_{b,t} + j_{c,t} k_b - j_b k_{c,t} - j_{b,t} k_c)(\sigma\sigma_t - \omega\omega_t) + \rho^2(j_b k_c \\
&- j_c k_b)(\sigma_t^2 + \sigma\sigma_{tt} - \omega_t^2 - \omega\omega_{tt}) + \mathrm{i}[(j_d k_a - j_a k_d)(\sigma_{tt}\omega - \sigma\omega_{tt}) + (j_d k_{a,t} \\
&+ j_{d,t} k_a - j_a k_{d,t} - j_{a,t} k_d)(\sigma_t\omega - \sigma\omega_t) + \rho(j_d k_{b,t} + j_{d,t} k_b - j_b k_{d,t} - j_{b,t} k_d) \\
&\cdot (\sigma_\rho\omega - \sigma\omega_\rho) + \rho(j_d k_b - j_b k_d)(\sigma_\rho\omega_t + \sigma_{t\rho}\omega - \sigma_t\omega_\rho - \sigma\omega_{t\rho})]\}
\end{aligned} \tag{D.3}$$

The function $F_{12} = F_d$ is

$$F_d = -[2q(\sigma^2-\omega^2)^2]^{-1}\{(2j_b k_{d,t} + 2j_{b,t} k_d - 2j_d k_{b,t} - 2j_{d,t} k_b + j_a k_{d,\rho\rho}$$

$$
\begin{aligned}
&+ 2j_{a,\rho}k_{d,\rho} + j_{a,\rho\rho}k_d - j_d k_{a,\rho\rho} - 2j_{d,\rho}k_{a,\rho} - j_{d,\rho\rho}k_a - 4m\sigma j_c)(\sigma^2 - \omega^2) \\
&+ 2(j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) + (j_a k_{d,\rho} + j_{a,\rho}k_d - j_d k_{a,\rho} - j_{d,\rho}k_a)(\sigma\sigma_\rho \\
&- \omega\omega_\rho) + (j_d k_a - j_a k_d)(\sigma_\rho^2 + \sigma\sigma_{\rho\rho} - \omega_\rho^2 - \omega\omega_{\rho\rho}) + (1/\rho)(j_a k_{d,\rho} + j_{a,\rho}k_d \\
&- j_d k_{a,\rho} - j_{d,\rho}k_a)(\sigma^2 - \omega^2) + (1/\rho)(j_d k_a - j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \rho(j_b k_{d,t\rho} \\
&+ j_{b,t}k_{d,\rho} + j_{b,\rho}k_{d,t} + j_{b,t\rho}k_d - j_d k_{b,t\rho} - j_{d,t}k_{b,\rho} - j_{d,\rho}k_{b,t} - j_{d,t\rho}k_b - 2m\sigma_\rho j_c \\
&- 2m\sigma j_{c,\rho})(\sigma^2 - \omega^2) + \rho(j_d k_{b,\rho} + j_{d,\rho}k_b - j_b k_{d,\rho} - j_{b,\rho}k_d)(\sigma\sigma_t - \omega\omega_t) \\
&+ 2\rho(j_b k_{d,t} + j_{b,t}k_d - j_d k_{b,t} - j_{d,t}k_b - 2m\sigma j_c)(\sigma\sigma_\rho - \omega\omega_\rho) + \rho(j_d k_b \\
&- j_b k_d)(\sigma_t\sigma_\rho + \sigma\sigma_{t\rho} - \omega_t\omega_\rho - \omega\omega_{t\rho}) + \mathrm{i}[2(j_c k_a - j_a k_c)(\sigma_t\omega - \sigma\omega_t) \\
&+ \rho(j_c k_{a,\rho} + j_{c,\rho}k_a - j_a k_{c,\rho} - j_{a,\rho}k_c)(\sigma_t\omega - \sigma\omega_t) + 3\rho(j_c k_b - j_b k_c)(\sigma_\rho\omega \\
&- \sigma\omega_\rho) + \rho(j_c k_a - j_a k_c)(\sigma_t\omega_\rho + \sigma_{t\rho}\omega - \sigma_\rho\omega_t - \sigma\omega_{t\rho}) + \rho^2(j_c k_b - j_b k_c) \\
&\cdot (\sigma_{\rho\rho}\omega - \sigma\omega_{\rho\rho}) + \rho^2(\sigma_\rho\omega - \sigma\omega_\rho)(j_c k_{b,\rho} + j_{c,\rho}k_b - j_b k_{c,\rho} - j_{b,\rho}k_c)] \\
&\cdot + (j_a k_d - j_d k_a)(j_b^2 + j_c^2 - k_b^2 - k_c^2) - \rho[(j_a k_b - j_b k_a)(j_b j_{d,\rho} - k_b k_{d,\rho}) \\
&+ (j_a k_c - j_c k_a)(j_c j_{d,\rho} - k_c k_{d,\rho}) + (j_b k_d - j_d k_b)(j_b j_{a,\rho} - k_b k_{a,\rho}) + (j_c k_d \\
&- j_d k_c)(j_c j_{a,\rho} - k_c k_{a,\rho}) + (j_d k_a - j_a k_d)(j_b j_{b,\rho} + j_c j_{c,\rho} - k_b k_{b,\rho} - k_c k_{c,\rho})]\} \\
&+ 2[q(\sigma^2 - \omega^2)^3]^{-1}(\sigma\sigma_\rho - \omega\omega_\rho)\{(j_{a,\rho}k_d + j_a k_{d,\rho} - j_d k_{a,\rho} - j_{d,\rho}k_a)(\sigma^2 - \omega^2) \\
&+ (j_d k_a - j_a k_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \rho(j_d k_b - j_b k_d)(\sigma\sigma_t - \omega\omega_t) + \rho(j_b k_{d,t} \\
&+ j_{b,t}k_d - j_d k_{b,t} - j_{d,t}k_b - 2m\sigma j_c)(\sigma^2 - \omega^2) + \mathrm{i}[\rho(j_c k_a - j_a k_c)(\sigma_t\omega - \sigma\omega_t) \\
&+ \rho^2(j_c k_b - j_b k_c)(\sigma_\rho\omega - \sigma\omega_\rho)]\}.
\end{aligned}
\tag{D.4}
$$

Lastly, the function in $F_{13} = xF_e$ and $F_{23} = yF_e$ is

$$
\begin{aligned}
F_e = &-2[q(\sigma^2 - \omega^2)^3]^{-1}(\sigma\sigma_\rho - \omega\omega_\rho)\{(j_c k_{a,\rho} + j_{c,\rho}k_a - j_a k_{c,\rho} - j_{a,\rho}k_c) \\
&\cdot (\sigma^2 - \omega^2) + (j_a k_c - j_c k_a)(\sigma\sigma_\rho - \omega\omega_\rho) + (2/\rho)(j_c k_a - j_a k_c - m\sigma j_d) \\
&\cdot (\sigma^2 - \omega^2) + \rho(j_c k_{b,t} + j_{c,t}k_b - j_b k_{c,t} - j_{b,t}k_c)(\sigma^2 - \omega^2) + \rho(j_b k_c - j_c k_b) \\
&\cdot (\sigma\sigma_t - \omega\omega_t) + \mathrm{i}[(j_d k_b - j_b k_d)(\sigma_\rho\omega - \sigma\omega_\rho) + (1/\rho)(j_d k_a - j_a k_d)(\sigma_t\omega \\
&- \sigma\omega_t)]\} + [2q(\sigma^2 - \omega^2)^2]^{-1}\{(2j_c k_{b,t} + 2j_{c,t}k_b - 2j_b k_{c,t} - 2j_{b,t}k_c + j_c k_{a,\rho\rho} \\
&+ 2j_{c,\rho}k_{a,\rho} + j_{c,\rho\rho}k_a - j_a k_{c,\rho\rho} - 2j_{a,\rho}k_{c,\rho} - j_{a,\rho\rho}k_c)(\sigma^2 - \omega^2) + 2(j_b k_c \\
&- j_c k_b)(\sigma\sigma_t - \omega\omega_t) + (j_c k_{a,\rho} + j_{c,\rho}k_a - j_a k_{c,\rho} - j_{a,\rho}k_c)(\sigma\sigma_\rho - \omega\omega_\rho) \\
&+ (j_a k_c - j_c k_a)(\sigma_\rho^2 + \sigma\sigma_{\rho\rho} - \omega_\rho^2 - \omega\omega_{\rho\rho}) + (1/\rho)(3j_c k_{a,\rho} + 3j_{c,\rho}k_a \\
&- 3j_a k_{c,\rho} - 3j_{a,\rho}k_c - 2m\sigma_\rho j_d - 2m\sigma j_{d,\rho})(\sigma^2 - \omega^2) + (1/\rho)(3j_c k_a \\
&- 3j_a k_c - 4m\sigma j_d)(\sigma\sigma_\rho - \omega\omega_\rho) + \rho(j_c k_{b,t\rho} + j_{c,t}k_{b,\rho} + j_{c,\rho}k_{b,t} + j_{c,t\rho}k_b \\
&- j_b k_{c,t\rho} - j_{b,t}k_{c,\rho} - j_{b,\rho}k_{c,t} - j_{b,t\rho}k_c)(\sigma^2 - \omega^2) + \rho(j_b k_{c,\rho} + j_{b,\rho}k_c - j_c k_{b,\rho} \\
&- j_{c,\rho}k_b)(\sigma\sigma_t - \omega\omega_t) + 2\rho(j_c k_{b,t} + j_{c,t}k_b - j_b k_{c,t} - j_{b,t}k_c)(\sigma\sigma_\rho - \omega\omega_\rho) \\
&+ \rho(j_b k_c - j_c k_b)(\sigma_t\sigma_\rho + \sigma\sigma_{t\rho} - \omega_t\omega_\rho - \omega\omega_{t\rho}) + \mathrm{i}[(j_d k_{b,\rho} + j_{d,\rho}k_b - j_b k_{d,\rho} \\
&- j_{b,\rho}k_d)(\sigma_\rho\omega - \sigma\omega_\rho) + (j_d k_b - j_b k_d)(\sigma_{\rho\rho}\omega - \sigma\omega_{\rho\rho}) + (1/\rho)(j_d k_{a,\rho} + j_{d,\rho}k_a \\
&- j_a k_{d,\rho} - j_{a,\rho}k_d)(\sigma_t\omega - \sigma\omega_t) + (1/\rho)(j_d k_b - j_b k_d)(\sigma_\rho\omega - \sigma\omega_\rho) \\
&+ (1/\rho)(j_d k_a - j_a k_d)(\sigma_{t\rho}\omega + \sigma_t\omega_\rho - \sigma_\rho\omega_t - \sigma\omega_{t\rho})]\}.
\end{aligned}
\tag{D.5}
$$

Appendix E

Derivation of the Belinfante Fierz identity

Here we supplement section 5.1.2 with a more detailed version of the derivation of (5.23). The four Fierz expansions containing the term we want to solve for, $[(\partial_\mu\overline{\psi})\gamma_\nu\psi - \overline{\psi}\gamma_\nu(\partial_\mu\psi)]$ are

$$
\begin{aligned}
j_\nu[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] &= \frac{\mathrm{i}}{3}(\partial_\mu j^\sigma)s_{\nu\sigma} - \frac{\mathrm{i}}{3}j^\sigma(\partial_\mu s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\omega)k_\nu - \frac{1}{3}\omega(\partial_\mu k_\nu) \\
&+ \frac{1}{3}\sigma[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] - \frac{\mathrm{i}}{3}{}^*s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] \\
&- \frac{\mathrm{i}}{3}k^\sigma[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi], \qquad \text{(E.1)}\\
j_\nu[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] &= \frac{\mathrm{i}}{3}(\partial_\mu j^\sigma){}^*s_{\nu\sigma} - \frac{\mathrm{i}}{3}j^\sigma(\partial_\mu{}^*s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\sigma)k_\nu - \frac{1}{3}\sigma(\partial_\mu k_\nu) \\
&+ \frac{1}{3}\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] - \frac{\mathrm{i}}{3}s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] \\
&- \frac{\mathrm{i}}{3}k^\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi], \qquad \text{(E.2)}\\
k_\nu(\partial_\mu\sigma) &= \frac{1}{3}\sigma(\partial_\mu k_\nu) - \frac{\mathrm{i}}{3}\partial_\mu(j^{\sigma*}s_{\nu\sigma}) + \frac{1}{3}j_\nu[\overline{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\psi] \\
&+ \frac{\mathrm{i}}{3}s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] - \frac{\mathrm{i}}{3}k^\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi] \\
&- \frac{1}{3}\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi], \qquad \text{(E.3)}\\
k_\nu(\partial_\mu\omega) &= \frac{1}{3}\omega(\partial_\mu k_\nu) - \frac{\mathrm{i}}{3}\partial_\mu(j^\sigma s_{\nu\sigma}) + \frac{1}{3}j_\nu[\overline{\psi}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\psi] \\
&+ \frac{\mathrm{i}}{3}{}^*s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi] - \frac{\mathrm{i}}{3}k^\sigma[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi] \\
&- \frac{1}{3}\sigma[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi]. \qquad \text{(E.4)}
\end{aligned}
$$

Combining these equations gives

$$
[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] = (\sigma\omega)^{-1}\Big(-\frac{\mathrm{i}}{2}(\partial_\mu j^\sigma)(\omega s_{\nu\sigma} + \sigma^* s_{\nu\sigma}) - k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)]
$$

$$+\frac{1}{2}(\partial_\mu k_\nu)(\sigma^2+\omega^2)+j_\nu\{\omega[\overline{\psi}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\psi]+\sigma[\overline{\psi}\gamma_5(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\psi]\}$$
$$\left.+\frac{\mathrm{i}}{2}(\sigma s_{\nu\sigma}+\omega{}^*s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\right), \tag{E.5}$$

which obviously requires more Fierz manipulation, since there are still spinor terms present. Using the Dirac identities (A.21)-(A.23), we obtain the additional Fierz expansions

$$\begin{aligned}&s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\\&=\frac{3\mathrm{i}}{5}\sigma(\partial_\mu k_\nu)-\frac{3\mathrm{i}}{5}(\partial_\mu\sigma)k_\nu+\frac{1}{5}j^\sigma(\partial_\mu{}^*s_{\nu\sigma})-\frac{1}{5}(\partial_\mu j^\sigma){}^*s_{\nu\sigma}+\frac{3\mathrm{i}}{5}j_\nu[\overline{\psi}\gamma_5(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\psi]\\&\quad-\frac{1}{5}k^\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi]+\frac{3\mathrm{i}}{5}\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi],\end{aligned}\tag{E.6}$$

$$\begin{aligned}&{}^*s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\\&=\frac{3\mathrm{i}}{5}\omega(\partial_\mu k_\nu)-\frac{3\mathrm{i}}{5}(\partial_\mu\omega)k_\nu+\frac{1}{5}j^\sigma(\partial_\mu s_{\nu\sigma})-\frac{1}{5}(\partial_\mu j^\sigma)s_{\nu\sigma}+\frac{3\mathrm{i}}{5}j_\nu[\overline{\psi}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\psi]\\&\quad-\frac{1}{5}k^\sigma[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi]+\frac{3\mathrm{i}}{5}\sigma[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi].\end{aligned}\tag{E.7}$$

Combining these expansions into the form they appear in (E.5), we get

$$\begin{aligned}&\frac{\mathrm{i}}{2}(\sigma s_{\nu\sigma}+\omega{}^*s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]=\frac{3}{10}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]\\&\quad-\frac{3}{10}(\sigma^2+\omega^2)(\partial_\mu k_\nu)-\frac{\mathrm{i}}{10}(\partial_\mu j^\sigma)(\sigma{}^*s_{\nu\sigma}+\omega s_{\nu\sigma})+\frac{\mathrm{i}}{10}j^\sigma[\sigma(\partial_\mu{}^*s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})]\\&\quad+\frac{3}{10}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]\\&\quad-\frac{\mathrm{i}}{10}k^\sigma\{\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi]+\omega[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi]\}\\&\quad-\frac{3}{5}\sigma\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi],\end{aligned}\tag{E.8}$$

which itself contains terms requiring further Fierz analysis. Using the Dirac identities (A.24)-(A.26), we find that the expansion of these terms is

$$\begin{aligned}&k^\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi]=\frac{1}{5}(\partial_\mu j^\sigma){}^*s_{\nu\sigma}-\frac{1}{5}j^\sigma(\partial_\mu{}^*s_{\nu\sigma})+\frac{3\mathrm{i}}{5}(\partial_\mu\sigma)k_\nu-\frac{3\mathrm{i}}{5}\sigma(\partial_\mu k_\nu)\\&\quad+\frac{3\mathrm{i}}{5}j_\nu[\overline{\psi}\gamma_5(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\psi]-\frac{1}{5}s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\\&\quad+\frac{3\mathrm{i}}{5}\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi],\end{aligned}\tag{E.9}$$

$$\begin{aligned}&k^\sigma[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi]=\frac{1}{5}(\partial_\mu j^\sigma)s_{\nu\sigma}-\frac{1}{5}j^\sigma(\partial_\mu s_{\nu\sigma})+\frac{3\mathrm{i}}{5}(\partial_\mu\omega)k_\nu-\frac{3\mathrm{i}}{5}\omega(\partial_\mu k_\nu)\\&\quad+\frac{3\mathrm{i}}{5}j_\nu[\overline{\psi}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\psi]-\frac{1}{5}{}^*s_{\nu\sigma}[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]\\&\quad+\frac{3\mathrm{i}}{5}\sigma[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi].\end{aligned}\tag{E.10}$$

Again, combining these terms into the form in which they appear in (E.8) gives

$$
\begin{aligned}
&-\frac{\mathrm{i}}{10}k^\sigma\{\sigma[\overline{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\sigma_{\nu\sigma}\psi]+\omega[\overline{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\sigma_{\nu\sigma}\psi]\}\\
&\quad=\frac{\mathrm{i}}{50}(\sigma s_{\nu\sigma}+\omega^* s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]+\frac{6}{50}\sigma\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi]\\
&\qquad-\frac{3}{50}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]-\frac{3}{50}(\sigma^2+\omega^2)(\partial_\mu k_\nu)\\
&\qquad+\frac{3}{50}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]+\frac{\mathrm{i}}{50}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})]-\frac{\mathrm{i}}{50}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma}),
\end{aligned}
\tag{E.11}
$$

which when substituting into (E.8) and rearranging, gives

$$
\begin{aligned}
&\frac{\mathrm{i}}{2}(\sigma s_{\nu\sigma}+\omega^* s_{\nu\sigma})[\overline{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\gamma^\sigma\psi]=-\frac{1}{2}\sigma\omega[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi]\\
&\qquad+\frac{1}{4}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]-\frac{3}{8}(\sigma^2+\omega^2)(\partial_\mu k_\nu)\\
&\qquad+\frac{3}{8}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]-\frac{\mathrm{i}}{8}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma})+\frac{\mathrm{i}}{8}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})],
\end{aligned}
\tag{E.12}
$$

a pure bilinear tensor expression. Now, using the Fierz identities derived in section 2.3

$$[\overline{\psi}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\psi]=-(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\omega+\mathrm{i}m^\nu(\partial_\mu n_\nu)\sigma], \tag{E.13}$$

$$[\overline{\psi}\gamma_5(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\psi]=-(\sigma^2-\omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\sigma+\mathrm{i}m^\nu(\partial_\mu n_\nu)\omega], \tag{E.14}$$

and combining them into the form in which they appear in (E.5), we get

$$
\begin{aligned}
&\omega[\overline{\psi}(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\psi]+\sigma[\overline{\psi}\gamma_5(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_5\psi]\\
&\qquad=-(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega],
\end{aligned}
\tag{E.15}
$$

which along with (E.12), can be substituted into (E.5) to give

$$
\begin{aligned}
&[\overline{\psi}\gamma_\nu(\partial_\mu\psi)-(\partial_\mu\overline{\psi})\gamma_\nu\psi]=(\sigma\omega)^{-1}\Big\{-\frac{1}{2}j_\nu(\sigma^2-\omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)+2\mathrm{i}m^\sigma(\partial_\mu n_\sigma)\sigma\omega]\\
&\qquad+\frac{1}{12}(\sigma^2+\omega^2)(\partial_\mu k_\nu)-\frac{5}{12}k_\nu[\sigma(\partial_\mu\sigma)+\omega(\partial_\mu\omega)]+\frac{\mathrm{i}}{12}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma})+\omega(\partial_\mu s_{\nu\sigma})]\\
&\qquad-\frac{5\mathrm{i}}{12}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma}+\omega s_{\nu\sigma})\Big\}.
\end{aligned}
\tag{E.16}
$$

This expression contains no explicit spinor terms, as required, but we can improve it by eliminating the rank-2 tensors $s_{\mu\nu}$ and $^*s_{\mu\nu}$, by using the Fierz identity

$$s^{\mu\nu}=(\sigma^2-\omega^2)^{-1}(\sigma\epsilon^{\mu\nu\rho\sigma}-\omega\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{E.17}$$

$${}^*s^{\mu\nu}=(\sigma^2-\omega^2)^{-1}(\omega\epsilon^{\mu\nu\rho\sigma}-\sigma\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{E.18}$$

where we define the partially antisymmetric object

$$\delta^{\mu\nu\rho\sigma}\equiv\mathrm{i}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}). \tag{E.19}$$

We also require the derivatives of these identities, which are

$$\partial_\mu s_{\nu\sigma} = \frac{\{[2\sigma\omega(\partial_\mu\omega) - (\sigma^2+\omega^2)(\partial_\mu\sigma)]\epsilon_{\nu\sigma\rho\epsilon} + [2\sigma\omega(\partial_\mu\sigma) - (\sigma^2+\omega^2)(\partial_\mu\omega)]\delta_{\nu\sigma\rho\epsilon}\}j^\rho k^\epsilon}{(\sigma^2-\omega^2)^2} + \frac{(\sigma\epsilon_{\nu\sigma\rho\epsilon} - \omega\delta_{\nu\sigma\rho\epsilon})[(\partial_\mu j^\rho)k^\epsilon + j^\rho(\partial_\mu k^\epsilon)]}{\sigma^2-\omega^2}, \tag{E.20}$$

$$\partial_\mu{}^* s_{\nu\sigma} = \frac{\{[-2\sigma\omega(\partial_\mu\sigma) + (\sigma^2+\omega^2)(\partial_\mu\omega)]\epsilon_{\nu\sigma\rho\epsilon} + [-2\sigma\omega(\partial_\mu\omega) + (\sigma^2+\omega^2)(\partial_\mu\sigma)]\delta_{\nu\sigma\rho\epsilon}\}j^\rho k^\epsilon}{(\sigma^2-\omega^2)^2} + \frac{(\omega\epsilon_{\nu\sigma\rho\epsilon} - \sigma\delta_{\nu\sigma\rho\epsilon})[(\partial_\mu j^\rho)k^\epsilon + j^\rho(\partial_\mu k^\epsilon)]}{\sigma^2-\omega^2}. \tag{E.21}$$

the rank-2 dependent terms in (E.16) become

$$\frac{\mathrm{i}}{12}j^\sigma[\sigma(\partial_\mu{}^* s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})] = \frac{1}{12}(\sigma^2-\omega^2)^{-1}\{2\sigma\omega k_\nu[\omega(\partial_\mu\sigma) - \sigma(\partial_\mu\omega)] + 2\mathrm{i}\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}j^\sigma(\partial_\mu j^\rho)k^\epsilon + j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2)\} - \frac{1}{12}(\partial_\mu k_\nu)(\sigma^2+\omega^2), \tag{E.22}$$

$$-\frac{5\mathrm{i}}{12}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma} + \omega s_{\nu\sigma}) = (\sigma^2-\omega^2)^{-1}\left\{\frac{5}{12}j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2+\omega^2) - \frac{5\mathrm{i}}{6}\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon - \frac{5}{12}k_\nu(\sigma^2+\omega^2)[\omega(\partial_\mu\omega) - \sigma(\partial_\mu\sigma)]\right\}, \tag{E.23}$$

giving us the final form of our identity

$$[\overline{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma_\nu\psi] = (\sigma^2-\omega^2)^{-1}\{k_\nu[\omega(\partial_\mu\sigma) - \sigma(\partial_\mu\omega)] - \mathrm{i}\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon - \mathrm{i}j_\nu m^\sigma(\partial_\mu n_\sigma)\}. \tag{E.24}$$

Appendix F

Fierz identities for J-K Lorentz vector current products

Here follow the general Fierz identities for Lorentz vector currents $J_i{}^\mu$, multiplied by the dual $K_j{}^\nu$. There are four different types of Pauli term arrangements: $a-b$, $0-a$, $a-0$ and $0-0$ which indicate the $i=0,1,2,3$ index in τ_i for the left and right terms in the product $J_i{}^\mu K_j{}^\nu$. We treat the $i=0$ and $i=a=b=1,2,3$ Pauli indices separately due to their slightly different algebraic properties. For each case, we write out the full Fierz expansion in spinor form, then expand Pauli and Dirac matrix products in terms of irreducible elements of the Pauli and Dirac algebra using identities from appendix A. The final Fierz identities are written in the non-Abelian current notation defined in section 7.1.

$a-b$ case:

$$
\begin{aligned}
J_a{}^\mu K_b{}^\nu &= \overline{\Psi}\tau_a\gamma^\mu(\Psi\overline{\Psi})\tau_b\gamma_5\gamma^\nu\Psi \\
&= -(1/8)J_i\overline{\Psi}\gamma_5\gamma^\mu\gamma^\nu\tau_a\tau^i\tau_b\Psi + (1/8)J_{i\sigma}\overline{\Psi}\gamma_5\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\tau_b\Psi \\
&\quad -(1/16)S_{i\sigma\epsilon}\overline{\Psi}\gamma_5\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu\tau_a\tau^i\tau_b\Psi \\
&\quad +(1/8)K_{i\sigma}\overline{\Psi}\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\tau_b\Psi + (1/8)K_i\overline{\Psi}\gamma^\mu\gamma^\nu\tau_a\tau^i\tau_b\Psi \\
&= -(1/8)J_0\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_{ab} + \mathrm{i}\epsilon_{ab}{}^d\tau_d]\Psi \\
&\quad -(1/8)J_c\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\tau_a\delta_b{}^c + \tau_b\delta_a{}^c - \tau^c\delta_{ab} - \mathrm{i}\epsilon_{ab}{}^c]\Psi \\
&\quad +(1/8)J_{0\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_{ab} + \mathrm{i}\epsilon_{ab}{}^d\tau_d]\Psi \\
&\quad +(1/8)J_{c\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\tau_a\delta_b{}^c + \tau_b\delta_a{}^c \\
&\quad -\tau^c\delta_{ab} - \mathrm{i}\epsilon_{ab}{}^c]\Psi \\
&\quad -(1/16)S_{0\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad +\mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda][\delta_{ab} + \mathrm{i}\epsilon_{ab}{}^d\tau_d]\Psi \\
&\quad -(1/16)S_{c\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad +\mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda][\tau_a\delta_b{}^c + \tau_b\delta_a{}^c - \tau^c\delta_{ab} - \mathrm{i}\epsilon_{ab}{}^c]\Psi \\
&\quad +(1/8)K_{0\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_{ab} + \mathrm{i}\epsilon_{ab}{}^d\tau_d]\Psi \\
&\quad +(1/8)K_{c\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\tau_a\delta_b{}^c + \tau_b\delta_a{}^c
\end{aligned}
$$

$$
\begin{aligned}
&- \tau^c\delta_{ab} - \mathrm{i}\epsilon_{ab}{}^c]\Psi \\
&+ (1/8)K_0\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_{ab} + \mathrm{i}\epsilon_{ab}{}^d\tau_d]\Psi \\
&+ (1/8)K_c\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\tau_a\delta_b{}^c + \tau_b\delta_a{}^c - \tau^c\delta_{ab} - \mathrm{i}\epsilon_{ab}{}^c]\Psi
\end{aligned}
$$

$$
\begin{aligned}
J_a{}^\mu K_b{}^\nu = (1/4)[&\mathrm{i}J_a{}^*S_b{}^{\mu\nu} + \mathrm{i}J_b{}^*S_a{}^{\mu\nu} - \mathrm{i}K_aS_b{}^{\mu\nu} - \mathrm{i}K_bS_a{}^{\mu\nu} + J_a{}^\mu K_b{}^\nu \\
&+ J_a{}^\nu K_b{}^\mu + J_b{}^\mu K_a{}^\nu + J_b{}^\nu K_a{}^\mu - J_{a\sigma}K_b{}^\sigma\eta^{\mu\nu} - J_{b\sigma}K_a{}^\sigma\eta^{\mu\nu} \\
&+ \delta_{ab}(\mathrm{i}J_0{}^*S_0{}^{\mu\nu} - \mathrm{i}J_c{}^*S^{c\mu\nu} - \mathrm{i}K_0S_0{}^{\mu\nu} + \mathrm{i}K_cS^{c\mu\nu} + J_0{}^\mu K_0{}^\nu \\
&+ J_0{}^\nu K_0{}^\mu - J_c{}^\mu K^{c\nu} - J_c{}^\nu K^{c\mu} - J_{0\sigma}K_0{}^\sigma\eta^{\mu\nu} + J_{c\sigma}K^{c\sigma}\eta^{\mu\nu})] \\
&+ (1/4)\epsilon_{ab}{}^c[-\mathrm{i}J_0K_c\eta^{\mu\nu} + \mathrm{i}K_0J_c\eta^{\mu\nu} + J_{c\sigma}J_{0\lambda}\epsilon^{\mu\nu\sigma\lambda} + K_{c\sigma}K_{0\lambda}\epsilon^{\mu\nu\sigma\lambda} \\
&+ (1/2)\mathrm{i}(-S_0{}^\mu{}_\sigma{}^*S_c{}^{\sigma\nu} - S_0{}^\nu{}_\sigma{}^*S_c{}^{\sigma\mu} + S_c{}^\mu{}_\sigma{}^*S_0{}^{\sigma\nu} + S_c{}^\nu{}_\sigma{}^*S_0{}^{\sigma\mu})]. \qquad \text{(F.1)}
\end{aligned}
$$

$a - 0$ case:

$$
\begin{aligned}
J_a{}^\mu K_0{}^\nu &= \overline{\Psi}\tau_a\gamma^\mu(\Psi\overline{\Psi})\gamma_5\gamma^\nu\Psi \\
&= -(1/8)J_i\overline{\Psi}\gamma_5\gamma^\mu\gamma^\nu\tau_a\tau^i\Psi + (1/8)J_{i\sigma}\overline{\Psi}\gamma_5\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\Psi \\
&\quad - (1/16)S_{i\sigma\epsilon}\overline{\Psi}\gamma_5\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu\tau_a\tau^i\Psi \\
&\quad + (1/8)K_{i\sigma}\overline{\Psi}\gamma^\mu\gamma^\sigma\gamma^\nu\tau_a\tau^i\Psi + (1/8)K_i\overline{\Psi}\gamma^\mu\gamma^\nu\tau_a\tau^i\Psi \\
&= -(1/8)J_0\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau_a\Psi - (1/8)J_c\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_a{}^c + \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&\quad + (1/8)J_{0\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau_a\Psi \\
&\quad + (1/8)J_{c\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_a{}^c + \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&\quad - (1/16)S_{0\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad + \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda]\tau_a\Psi \\
&\quad - (1/16)S_{c\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad + \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda][\delta_a{}^c + \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&\quad + (1/8)K_{0\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau_a\Psi \\
&\quad + (1/8)K_{c\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_a{}^c + \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&\quad + (1/8)K_0\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau_a\Psi \\
&\quad + (1/8)K_c\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_a{}^c + \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi
\end{aligned}
$$

$$
\begin{aligned}
J_a{}^\mu K_0{}^\nu = (1/4)[&\mathrm{i}J_0{}^*S_a{}^{\mu\nu} + \mathrm{i}J_a{}^*S_0{}^{\mu\nu} - \mathrm{i}K_0S_a{}^{\mu\nu} - \mathrm{i}K_aS_0{}^{\mu\nu} + J_0{}^\mu K_a{}^\nu \\
&+ J_0{}^\nu K_a{}^\mu + J_a{}^\mu K_0{}^\nu + J_a{}^\nu K_0{}^\mu - J_{0\sigma}K_a{}^\sigma\eta^{\mu\nu} - J_{a\sigma}K_0{}^\sigma\eta^{\mu\nu}] \\
&- (1/4)\epsilon_a{}^{cd}[\mathrm{i}J_cK_d\eta^{\mu\nu} + (1/2)J_{c\sigma}J_{d\lambda}\epsilon^{\mu\nu\sigma\lambda} + (1/2)K_{c\sigma}K_{d\lambda}\epsilon^{\mu\nu\sigma\lambda} \\
&- (1/2)\mathrm{i}(S_d{}^\mu{}_\sigma{}^*S_c{}^{\sigma\nu} + S_d{}^\nu{}_\sigma{}^*S_c{}^{\sigma\mu})] \qquad \text{(F.2)}
\end{aligned}
$$

$0 - a$ case:

$$
\begin{aligned}
J_0{}^\mu K_a{}^\nu &= \overline{\Psi}\gamma^\mu(\Psi\overline{\Psi})\tau_a\gamma_5\gamma^\nu\Psi \\
&= -(1/8)J_i\overline{\Psi}\gamma_5\gamma^\mu\gamma^\nu\tau^i\tau_a\Psi + (1/8)J_{i\sigma}\overline{\Psi}\gamma_5\gamma^\mu\gamma^\sigma\gamma^\nu\tau^i\tau_a\Psi
\end{aligned}
$$

$$
\begin{aligned}
&- (1/16)S_{i\sigma\epsilon}\overline{\Psi}\gamma_5\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu\tau^i\tau_a\Psi \\
&+ (1/8)K_{i\sigma}\overline{\Psi}\gamma^\mu\gamma^\sigma\gamma^\nu\tau^i\tau_a\Psi + (1/8)K_i\overline{\Psi}\gamma^\mu\gamma^\nu\tau^i\tau_a\Psi \\
= &- (1/8)J_0\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau_a\Psi - (1/8)J_c\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_a{}^c - \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&+ (1/8)J_{0\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau_a\Psi \\
&+ (1/8)J_{c\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_a{}^c - \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&- (1/16)S_{0\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&+ \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda]\tau_a\Psi \\
&- (1/16)S_{c\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&+ \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda][\delta_a{}^c - \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&+ (1/8)K_{0\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau_a\Psi \\
&+ (1/8)K_{c\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda][\delta_a{}^c - \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi \\
&+ (1/8)K_0\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau_a\Psi \\
&+ (1/8)K_c\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}][\delta_a{}^c - \mathrm{i}\epsilon_a{}^{cd}\tau_d]\Psi
\end{aligned}
$$

$$
\begin{aligned}
J_0{}^\mu K_a{}^\nu = {} & (1/4)[\mathrm{i}J_0{}^*S_a{}^{\mu\nu} + \mathrm{i}J_a{}^*S_0{}^{\mu\nu} - \mathrm{i}K_0S_a{}^{\mu\nu} - \mathrm{i}K_aS_0{}^{\mu\nu} + J_0{}^\mu K_a{}^\nu \\
&+ J_0{}^\nu K_a{}^\mu + J_a{}^\mu K_0{}^\nu + J_a{}^\nu K_0{}^\mu - J_{0\sigma}K_a{}^\sigma\eta^{\mu\nu} - J_{a\sigma}K_0{}^\sigma\eta^{\mu\nu}] \\
&+ (1/4)\epsilon_a{}^{cd}[\mathrm{i}J_cK_d\eta^{\mu\nu} + (1/2)J_{c\sigma}J_{d\lambda}\epsilon^{\mu\nu\sigma\lambda} + (1/2)K_{c\sigma}K_{d\lambda}\epsilon^{\mu\nu\sigma\lambda} \\
&- (1/2)\mathrm{i}(S_d{}^\mu{}_\sigma{}^*S_c{}^{\sigma\nu} + S_d{}^\nu{}_\sigma{}^*S_c{}^{\sigma\mu})]
\end{aligned}
\tag{F.3}
$$

$0-0$ case:

$$
\begin{aligned}
J_0{}^\mu K_0{}^\nu &= \overline{\Psi}\gamma^\mu(\Psi\overline{\Psi})\gamma_5\gamma^\nu\Psi \\
&= - (1/8)J_i\overline{\Psi}\gamma_5\gamma^\mu\gamma^\nu\tau^i\Psi + (1/8)J_{i\sigma}\overline{\Psi}\gamma_5\gamma^\mu\gamma^\sigma\gamma^\nu\tau^i\Psi \\
&\quad - (1/16)S_{i\sigma\epsilon}\overline{\Psi}\gamma_5\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu\tau^i\Psi \\
&\quad + (1/8)K_{i\sigma}\overline{\Psi}\gamma^\mu\gamma^\sigma\gamma^\nu\tau^i\Psi + (1/8)K_i\overline{\Psi}\gamma^\mu\gamma^\nu\tau^i\Psi \\
&= - (1/8)J_0\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\Psi - (1/8)J_c\overline{\Psi}\gamma_5[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau^c\Psi \\
&\quad + (1/8)J_{0\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\Psi \\
&\quad + (1/8)J_{c\sigma}\overline{\Psi}\gamma_5[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau^c\Psi \\
&\quad - (1/16)S_{0\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad + \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda]\Psi \\
&\quad - (1/16)S_{c\sigma\epsilon}\overline{\Psi}\gamma_5[\mathrm{i}\eta^{\epsilon\nu}\eta^{\mu\sigma} - \mathrm{i}\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 \\
&\quad + \mathrm{i}\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda]\tau^c\Psi \\
&\quad + (1/8)K_{0\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\Psi \\
&\quad + (1/8)K_{c\sigma}\overline{\Psi}[\eta^{\mu\sigma}\gamma^\nu + \eta^{\sigma\nu}\gamma^\mu - \eta^{\mu\nu}\gamma^\sigma - \mathrm{i}\epsilon^{\mu\sigma\nu\lambda}\gamma_5\gamma_\lambda]\tau^c\Psi \\
&\quad + (1/8)K_0\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\Psi + (1/8)K_c\overline{\Psi}[\eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}]\tau^c\Psi
\end{aligned}
$$

$$
\begin{aligned}
J_0{}^\mu K_0{}^\nu = {} & (1/4)[\mathrm{i}J_0{}^*S_0{}^{\mu\nu} + \mathrm{i}J_c{}^*S^{c\mu\nu} - \mathrm{i}K_0S_0{}^{\mu\nu} - \mathrm{i}K_cS^{c\mu\nu} + J_0{}^\mu K_0{}^\nu + J_0{}^\nu K_0{}^\mu \\
&+ J_c{}^\mu K^{c\nu} + J_c{}^\nu K^{c\mu} - J_{0\sigma}K_0{}^\sigma\eta^{\mu\nu} - J_{c\sigma}K^{c\sigma}\eta^{\mu\nu}]
\end{aligned}
\tag{F.4}
$$

Appendix G

Fierz Identities for Rank-2 Skew Tensor Currents

This appendix contains a more detailed version of the derivation of the Fierz identities for ${S_0}^{\mu\nu}$, ${{}^*S_0}^{\mu\nu}$, ${S_a}^{\mu\nu}$ and ${{}^*S_a}^{\mu\nu}$. Let us first derive expressions for ${S_0}^{\mu\nu}$ and ${{}^*S_0}^{\mu\nu}$, using the Fierz identities for JK vector current products derived in appendix F. Consider the following expression, obtained from the Fierz identity (F.4) and subtracting the Pauli trace of (F.1):

$$\begin{aligned}{J_0}^{\mu}{K_0}^{\nu} - {J_a}^{\nu}K^{a\mu} &= \mathrm{i}J_0{{}^*S_0}^{\mu\nu} - \mathrm{i}K_0{S_0}^{\mu\nu} - (1/2){J_0}^{\mu}{K_0}^{\nu} - (1/2){J_0}^{\nu}{K_0}^{\mu} \\ &+ (1/2){J_a}^{\mu}K^{a\nu} + (1/2){J_a}^{\nu}K^{a\mu} + (1/2)J_{0\sigma}{K_0}^{\sigma}\eta^{\mu\nu} - (1/2)J_{a\sigma}K^{a\sigma}\eta^{\mu\nu}. \quad \text{(G.1)}\end{aligned}$$

Note that we have used the Fierz forms that have no stand-alone Pauli vector triplet indices $a = 1, 2, 3$. Similarly, using the same Fierz identities we can also form

$$\begin{aligned}{J_a}^{\mu}K^{a\nu} - {J_0}^{\nu}{K_0}^{\mu} &= \mathrm{i}J_0{{}^*S_0}^{\mu\nu} - \mathrm{i}K_0{S_0}^{\mu\nu} + (1/2){J_0}^{\mu}{K_0}^{\nu} + (1/2){J_0}^{\nu}{K_0}^{\mu} \\ &- (1/2){J_a}^{\mu}K^{a\nu} - (1/2){J_a}^{\nu}K^{a\mu} - (1/2)J_{0\sigma}{K_0}^{\sigma}\eta^{\mu\nu} + (1/2)J_{a\sigma}K^{a\sigma}\eta^{\mu\nu}. \quad \text{(G.2)}\end{aligned}$$

Adding these two equations, we get

$${J_i}^{\mu}K^{i\nu} - {J_i}^{\nu}K^{i\mu} = 2\mathrm{i}(J_0{{}^*S_0}^{\mu\nu} - K_0{S_0}^{\mu\nu}) \quad \text{(G.3)}$$

which is antisymmetric in μ, ν. Another Fierz expression which is antisymmetric in μ, ν, is

$$\begin{aligned}\epsilon^{\mu\nu\rho\kappa}J_{0\rho}K_{0\kappa} &= (1/4)\epsilon^{\mu\nu\rho\kappa}[\mathrm{i}J_0{}^*S_{0\rho\kappa} + \mathrm{i}J_a{}^*{S^a}_{\rho\kappa} - \mathrm{i}K_0S_{0\rho\kappa} - \mathrm{i}K_a{S^a}_{\rho\kappa} + J_{0\rho}K_{0\kappa} \\ &\quad + J_{0\kappa}K_{0\rho} + J_{a\rho}{K^a}_{\kappa} + J_{a\kappa}{K^a}_{\rho} - J_{0\sigma}{K_0}^{\sigma}\eta_{\rho\kappa} - J_{a\sigma}K^{a\sigma}\eta_{\rho\kappa}] \\ &= -(1/8)(\epsilon^{\mu\nu\rho\kappa}\epsilon_{\rho\kappa\delta\tau})J_0{S_0}^{\delta\tau} - (1/8)(\epsilon^{\mu\nu\rho\kappa}\epsilon_{\rho\kappa\delta\tau})J_aS^{a\delta\tau} \\ &\quad - (1/2)K_0{{}^*S_0}^{\mu\nu} - (1/2)K_a{}^*S^{a\mu\nu} \\ &= (1/2)J_0{S_0}^{\mu\nu} + (1/2)J_aS^{a\mu\nu} - (1/2)K_0{{}^*S_0}^{\mu\nu} - (1/2)K_a{}^*S^{a\mu\nu} \quad \text{(G.4)}\end{aligned}$$

where we have first canceled out all of the terms from (F.4) symmetric in μ, ν, then applied the Levi-Civita double contraction identity

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho'\sigma'} = 2({\delta^{\rho}}_{\sigma'}{\delta^{\sigma}}_{\rho'} - {\delta^{\rho}}_{\rho'}{\delta^{\sigma}}_{\sigma'}) \quad \text{(G.5)}$$

in addition to the definition (7.13) to convert the S terms to ${}^*S$, and vice-versa. The other term we can contract with the Levi-Civita tensor, with no stand-alone Pauli vector triplet indices is the trace over the Pauli index a. So after (again) canceling all symmetric terms from (F.1) and applying the Levi-Civita contraction identity, we get

$$\epsilon^{\mu\nu\rho\kappa}J_{a\rho}K^{a}{}_{\kappa} = (3/2)J_0S_0{}^{\mu\nu} - (1/2)J_aS^{a\mu\nu} - (3/2)K_0{}^*S_0{}^{\mu\nu} + (1/2)K_a{}^*S^{a\mu\nu} \quad \text{(G.6)}$$

Taking the sum of (G.4) and (G.6), we obtain

$$\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^{i}{}_{\kappa} = 2(J_0S_0{}^{\mu\nu} - K_0{}^*S_0{}^{\mu\nu}). \quad \text{(G.7)}$$

Now take the combination of (G.3) and (G.7)

$$\begin{aligned} &J_0\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^{i}{}_{\kappa} - \mathrm{i}K_0(J_i{}^{\mu}K^{i\nu} - J_i{}^{\nu}K^{i\mu}) \\ &\quad = 2(J_0^2S_0{}^{\mu\nu} - J_0K_0{}^*S_0{}^{\mu\nu} + J_0K_0{}^*S_0{}^{\mu\nu} - K_0^2S_0{}^{\mu\nu}). \end{aligned} \quad \text{(G.8)}$$

Canceling the middle two terms on the right-hand side and rearranging, we end up with one of our identities

$$S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[J_0\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^{i}{}_{\kappa} - \mathrm{i}K_0(J_i{}^{\mu}K^{i\nu} - J_i{}^{\nu}K^{i\mu})]. \quad \text{(G.9)}$$

By comparison, the Abelian version of the identity is [10], [27],

$$s^{\mu\nu} = (\sigma^2 - \omega^2)^{-1}[\sigma\epsilon^{\mu\nu\rho\kappa}j_\rho k_\kappa - \mathrm{i}\omega(j^\mu k^\nu - j^\nu k^\mu)], \quad \text{(G.10)}$$

where we have used an alternate definition for the pseudoscalar, $\omega \equiv \overline{\psi}\gamma_5\psi$, as opposed to the often used $\overline{\psi}\mathrm{i}\gamma_5\psi$. Now to calculate the dual, take an alternate combination of (G.3) and (G.7)

$$\begin{aligned} &K_0\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^{i}{}_{\kappa} - \mathrm{i}J_0(J_i{}^{\mu}K^{i\nu} - J_i{}^{\nu}K^{i\mu}) \\ &\quad = 2(J_0K_0S_0{}^{\mu\nu} - K_0^{2*}S_0{}^{\mu\nu} + J_0^{2*}S_0{}^{\mu\nu} - J_0K_0S_0{}^{\mu\nu}) \end{aligned} \quad \text{(G.11)}$$

which immediately leads to

$${}^*S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[K_0\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^{i}{}_{\kappa} - \mathrm{i}J_0(J_i{}^{\mu}K^{i\nu} - J_i{}^{\nu}K^{i\mu})]. \quad \text{(G.12)}$$

Let us now derive an expression for $S_a{}^{\mu\nu}$. We start by forming an expression from the JK Lorentz vector current product Fierz identities, with a single Pauli vector triplet index, (F.2) and (F.3),

$$\begin{aligned} J_a{}^{\mu}K_0{}^{\nu} + J_0{}^{\mu}K_a{}^{\nu} = (1/2)[&\mathrm{i}J_0{}^*S_a{}^{\mu\nu} + \mathrm{i}J_a{}^*S_0{}^{\mu\nu} - \mathrm{i}K_0S_a{}^{\mu\nu} - \mathrm{i}K_aS_0{}^{\mu\nu} + J_0{}^{\mu}K_a{}^{\nu} \\ &+ J_0{}^{\nu}K_a{}^{\mu} + J_a{}^{\mu}K_0{}^{\nu} + J_a{}^{\nu}K_0{}^{\mu} - J_{0\sigma}K_a{}^{\sigma}\eta^{\mu\nu} - J_{a\sigma}K_0{}^{\sigma}\eta^{\mu\nu}], \end{aligned} \quad \text{(G.13)}$$

which eliminates the Pauli Levi-Civita contracted terms that are in the separate $a-0$ and $0-a$ cases, as they are identical and of opposite sign in each. Subtracting (G.13) from itself, but with the μ,ν terms flipped, will cancel all of the terms symmetric in μ,ν:

$$(J_a{}^{\mu}K_0{}^{\nu} + J_0{}^{\mu}K_a{}^{\nu}) - (J_a{}^{\nu}K_0{}^{\mu} + J_0{}^{\nu}K_a{}^{\mu})$$

$$= \mathrm{i}J_0{}^*S_a{}^{\mu\nu} + \mathrm{i}J_a{}^*S_0{}^{\mu\nu} - \mathrm{i}K_0S_a{}^{\mu\nu} - \mathrm{i}K_aS_0{}^{\mu\nu}. \tag{G.14}$$

Another way to get rid of the terms symmetric in μ, ν is to contract (G.13) with $\epsilon^{\mu\nu\rho\kappa}$:

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) \\
&\quad = (1/2)\epsilon^{\mu\nu\rho\kappa}[\mathrm{i}J_0{}^*S_{a\rho\kappa} + \mathrm{i}J_a{}^*S_{0\rho\kappa} - \mathrm{i}K_0S_{a\rho\kappa} - \mathrm{i}K_aS_{0\rho\kappa}] \\
&\quad = -(1/4)(\epsilon^{\mu\nu\rho\kappa}\epsilon_{\rho\kappa\delta\tau})J_0S_a{}^{\delta\tau} - (1/4)(\epsilon^{\mu\nu\rho\kappa}\epsilon_{\rho\kappa\delta\tau})J_aS_0{}^{\delta\tau} - K_0{}^*S_a{}^{\mu\nu} - K_a{}^*S_0{}^{\mu\nu} \\
&\quad = -(1/2)(\delta^\mu{}_\tau\delta^\nu{}_\delta - \delta^\mu{}_\delta\delta^\nu{}_\tau)J_0S_a{}^{\delta\tau} - (1/2)(\delta^\mu{}_\tau\delta^\nu{}_\delta - \delta^\mu{}_\delta\delta^\nu{}_\tau)J_aS_0{}^{\delta\tau} - K_0{}^*S_a{}^{\mu\nu} \\
&\qquad - K_a{}^*S_0{}^{\mu\nu},
\end{aligned} \tag{G.15}$$

where we have used the Levi-Civita tensor double contraction (G.5). Simplifying gives

$$\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) = J_0S_a{}^{\mu\nu} + J_aS_0{}^{\mu\nu} - K_0{}^*S_a{}^{\mu\nu} - K_a{}^*S_0{}^{\mu\nu}. \tag{G.16}$$

Now take the combination of (G.14) and (G.16):

$$\begin{aligned}
&J_0\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) - \mathrm{i}K_0[(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) - (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu)] \\
&\quad = J_0^2S_a{}^{\mu\nu} + J_0J_aS_0{}^{\mu\nu} - J_0K_0{}^*S_a{}^{\mu\nu} - J_0K_a{}^*S_0{}^{\mu\nu} + K_0J_0{}^*S_a{}^{\mu\nu} + K_0J_a{}^*S_0{}^{\mu\nu} \\
&\qquad - K_0^2S_a{}^{\mu\nu} - K_0K_aS_0{}^{\mu\nu} \\
&\quad = (J_0^2 - K_0^2)S_a{}^{\mu\nu} + (J_0J_a - K_0K_a)S_0{}^{\mu\nu} + (K_0J_a - J_0K_a){}^*S_0{}^{\mu\nu}.
\end{aligned} \tag{G.17}$$

Let us rewrite the second and third terms on the right-hand side, by substituting the $S_0{}^{\mu\nu}$ and ${}^*S_0{}^{\mu\nu}$ identities, (G.9) and (G.12):

$$\begin{aligned}
(J_0J_a - K_0K_a)S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[&(J_0^2J_a - J_0K_0K_a)\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa \\
&+ \mathrm{i}(K_0^2K_a - J_0K_0J_a)(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.18}$$

Likewise, for the dual we have:

$$\begin{aligned}
(K_0J_a - J_0K_a){}^*S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1}[&(K_0^2J_a - J_0K_0K_a)\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa \\
&+ \mathrm{i}(J_0^2K_a - J_0K_0J_a)(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.19}$$

Summing these two equations together gives

$$\begin{aligned}
&(J_0J_a - K_0K_a)S_0{}^{\mu\nu} + (K_0J_a - J_0K_a){}^*S_0{}^{\mu\nu} \\
&\quad = \frac{J_0^2 + K_0^2}{2(J_0^2 - K_0^2)}[J_a\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa + \mathrm{i}K_a(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})] \\
&\qquad - \frac{J_0K_0}{J_0^2 - K_0^2}[K_a\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa + \mathrm{i}J_a(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.20}$$

Substituting this into (G.17) and rearranging, we finally get

$$\begin{aligned}
S_a{}^{\mu\nu} = (J_0^2 - K_0^2)^{-1}\{&J_0\epsilon^{\mu\nu\rho\kappa}(J_{a\rho}K_{0\kappa} + J_{0\rho}K_{a\kappa}) - \mathrm{i}K_0[(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) \\
&- (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu)]\} - \frac{J_0^2 + K_0^2}{2(J_0^2 - K_0^2)^2}[J_a\epsilon^{\mu\nu\rho\kappa}J_{i\rho}K^i{}_\kappa + \mathrm{i}K_a(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})]
\end{aligned}$$

$$+ \frac{J_0 K_0}{(J_0^2 - K_0^2)^2}[K_a \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa + \mathrm{i} J_a (J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})]. \tag{G.21}$$

Now let us derive the dual, ${}^*S_a{}^{\mu\nu}$ by considering the alternate combination:

$$\begin{aligned}
K_0 \epsilon^{\mu\nu\rho\kappa}&(J_{a\rho} K_{0\kappa} + J_{0\rho} K_{a\kappa}) - \mathrm{i} J_0 [(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) - (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu)] \\
&= J_0 K_0 S_a{}^{\mu\nu} + K_0 J_a S_0{}^{\mu\nu} - K_0^2 {}^*S_a{}^{\mu\nu} - K_0 K_a {}^*S_0{}^{\mu\nu} + J_0^2 {}^*S_a{}^{\mu\nu} + J_0 J_a {}^*S_0{}^{\mu\nu} \\
&\qquad - J_0 K_0 S_a{}^{\mu\nu} - J_0 K_a S_0{}^{\mu\nu} \\
&= (J_0^2 - K_0^2) {}^*S_a{}^{\mu\nu} + (K_0 J_a - J_0 K_a) S_0{}^{\mu\nu} + (J_0 J_a - K_0 K_a) {}^*S_0{}^{\mu\nu}.
\end{aligned} \tag{G.22}$$

Again, we rewrite the second and third terms, by substituting identities (G.9) and (G.12):

$$\begin{aligned}
(K_0 J_a - J_0 K_a) S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1} &[(J_0 K_0 J_a - J_0^2 K_a) \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa \\
&+ \mathrm{i}(J_0 K_0 K_a - K_0^2 J_a)(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.23}$$

The dual term is

$$\begin{aligned}
(J_0 J_a - K_0 K_a) {}^*S_0{}^{\mu\nu} = (1/2)(J_0^2 - K_0^2)^{-1} &[(J_0 K_0 J_a - K_0^2 K_a) \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa \\
&+ \mathrm{i}(J_0 K_0 K_a - J_0^2 J_a)(J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.24}$$

Summing the two:

$$\begin{aligned}
(K_0 J_a &- J_0 K_a) S_0{}^{\mu\nu} + (J_0 J_a - K_0 K_a) {}^*S_0{}^{\mu\nu} \\
&= \frac{J_0 K_0}{J_0^2 - K_0^2}[J_a \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa + \mathrm{i} K_a (J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})] \\
&\quad - \frac{(J_0^2 + K_0^2)}{2(J_0^2 - K_0^2)}[K_a \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa + \mathrm{i} J_a (J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.25}$$

Finally, after substituting this into (G.22) and rearranging, we get:

$$\begin{aligned}
{}^*S_a{}^{\mu\nu} &= (J_0^2 - K_0^2)^{-1} \{ K_0 \epsilon^{\mu\nu\rho\kappa} (J_{a\rho} K_{0\kappa} + J_{0\rho} K_{a\kappa}) - \mathrm{i} J_0 [(J_a{}^\mu K_0{}^\nu + J_0{}^\mu K_a{}^\nu) \\
&- (J_a{}^\nu K_0{}^\mu + J_0{}^\nu K_a{}^\mu)] \} + \frac{J_0^2 + K_0^2}{2(J_0^2 - K_0^2)^2}[K_a \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa + \mathrm{i} J_a (J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})] \\
&- \frac{J_0 K_0}{(J_0^2 - K_0^2)^2}[J_a \epsilon^{\mu\nu\rho\kappa} J_{i\rho} K^i{}_\kappa + \mathrm{i} K_a (J_i{}^\mu K^{i\nu} - J_i{}^\nu K^{i\mu})].
\end{aligned} \tag{G.26}$$

Comparing this identity with the identity for $S_a{}^{\mu\nu}$, (G.21), we can make the observations that the first terms are the same in each, but with the J_0 and K_0 terms interchanged (similarly to the $S_0{}^{\mu\nu}$ and ${}^*S_0{}^{\mu\nu}$ comparison). Likewise, the second and third terms of both identities are the same, but with the J_a and K_a terms interchanged, *along with the signs*, which are flipped with respect to each other.

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